VOLUME XLVIII NUMBER FOUR

The Mathematics Teacher

APRIL 1955

The official journal of
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Mathematics Teacher *is a journal of The National Council of Teachers of Mathematics devoted to the interests of mathematics teachers in the Junior High Schools, Senior High Schools, Junior Colleges and Teacher Education Colleges.*

All editorial correspondence, including books for review, should be addressed to the Editor.

All other correspondence should be addressed to

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street N. W., Washington 6, D. C.

Printed at Menasha, Wisconsin, U.S.A. Entered as second-class matter at the post office at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412 P. L. & R., authorized March 1, 1930. Printed in U.S.A.

The Mathematics Teacher

volume XLVIII, number 4 April 1955

MEMBERSHIP FEE: $3.00 per year for individual membership. All members of *The National Council of Teachers of Mathematics* receive eight issues of THE MATHEMATICS TEACHER, October through May. Institutional Memberships $5.00 per year. Student Membership: $1.50 first year. Single copies 50 cents. Remittance should be made payable to *The National Council of Teachers of Mathematics* and sent to the Washington Office. Canadian postage, 25 cents per year. All other foreign postage 50 cents per year.

Some educational problems of significance to engineering colleges[1]

CARROLL V. NEWSOM, *Associate Commissioner of Education, The University of the State of New York, Albany, New York.*

MY RESPONSIBILITIES for higher and professional education in the state of New York include the maintenance of satisfactory standards of education for those students who desire to enter the seventeen professions recognized by the statutes of the state. This fact, combined with the experience of a quarter of a century of teaching college mathematics on both the undergraduate and graduate levels, including many classes that were composed chiefly of engineering students, has caused me to become genuinely concerned with many educational problems. In fact, two years ago I organized, in the state of New York, a High School-College Articulation Committee to attempt to resolve some of the issues with which we are faced. This article is concerned essentially with some of the findings that have resulted from that committee's deliberations, along with some of my personal opinions.

An examination of the situation that exists on the high school level forces one to the conclusion that there are problems in the field of public education that are much more complicated than many of us may have suspected. It would appear that our educational effort as it pertains to the high schools has succeeded admirably in some directions but has been far from successful in others. First, therefore, let us consider some of the problems that confront the public schools.

In 1945, the Harvard Committee on the Objectives of a General Education in a Free Society published a report which, it is claimed, "presents a view of the total American educational scene." In this report appears the following statement:

> In the seventy years between 1870 and 1940 the population slightly more than tripled. But in 1870 some 80,000 students were enrolled in secondary schools and 60,000 in colleges, whereas by 1940, 7,000,000 were enrolled in the former and 1,500,000 in the latter. Thus, while the general population was increasing three times over, the enrollment of high schools was being multiplied about ninety times and that of colleges about thirty times. And the end is not yet. Even now one young person in six fails to reach high school, and half of those who enter drop out before the end.

The report goes on to state that the ninetyfold increase in high school enrollment has not meant simply a ninetyfold multiplication of the old plan and kind of schooling. Whereas in 1870 three-fourths of those who attended high school went on to college, now three-fourths of all high school students look forward to employment at the conclusion of their secondary studies. The Harvard report raises the important question:

> How, given this new character and rule of the high school, can the interests of the three-fourths who go on to active life be reconciled with the equally just interests of the one-fourth who go on to further education?

I have discovered that experts at the secondary level admit frankly that they have not succeeded in providing a satisfactory answer to this question.

Our public schools are filled to overflowing with great numbers of students of varied interests and abilities; this variety may be such as to virtually prevent a

[1] Invited address given before the summer meeting of the A.S.E.E., June 1952.

sound program in instruction in terms of the needs of the students. It is inevitable that some solution must be developed that will involve a differentiation of students in terms of interest and ability, but it appears that such a remedy is not easily accomplished. A major difficulty results from the fact that there seem to be no adequate criteria for the determination of ability; my own experience as a teacher confirms such a belief. At one time I actually created a study to discover why students who rated in the upper ten per cent of the entrance examinations in mathematics never seemed to be able to become mathematics majors; such students did very well during the freshman year, but most of them fell by the wayside long before their senior year. Those who ultimately became candidates for the Ph.D. degree were usually in the middle brackets of intelligence, as determined by the tests that were given them at the time of college entrance. I understand that psychologists are in considerable disagreement upon the significance of intelligence tests. There are other problems involved in this suggestion of differentiation; the element of drive and ambition on the part of the student cannot be ignored. We have found in the state of New York that less than half of the students whose grades place them in the upper quarter of the high school graduating class ever go on to college. Economic factors have been advanced as the reason for this phenomenon but some students of the subject emphasize, as a partial explanation, the lack of motivation on the part of many competent youngsters. Also, we cannot ignore the importance of parental prejudices in any plan for differentiation. Many parents are determined that their children should be placed in the so-called college sequence in high school, when it is available, irrespective of the children's ability. I have been told by some high school principals that most parents completely reject differentiation of students in terms of ability. The creation of two or more sequences of courses in high school also encounters difficulties of a financial nature; it appears that many of the smaller high schools cannot afford this kind of multiple-track program. I have suggested frequently that it may be necessary to develop correspondence courses for high school students in order to provide special opportunities for students of ability and unusual aptitudes. I can well understand the tremendous problems involved in a program of mass education such as we have in this country; nevertheless, I believe it is urgent that there be continuing efforts toward the provision of special educational opportunities for certain students.

To continue our analysis of pre-college education, it is interesting to observe the striking manner in which the curriculum in the high school has changed during the past half century. In considering this change, the late Commissioner Spaulding of my own organization wrote:

> The subjects studied have changed. They have changed in part simply because there are more things in the world now to learn about than there used to be: aviation, electronics, the Soviet Union, the United Nations, the economics of prosperity and depression, the atomic bomb, to mention only a few. The subjects taught have changed also because they now include many things that were formerly not necessary for a high school curriculum because they were not an essential part of preparation for life: industrial arts, home economics, agriculture, citizenship, group living.

It is only reasonable to expect that this great expansion in the content of the high school program would have an impact upon the emphasis that is given to those subjects that many of us regard as fundamental, especially to a curriculum such as engineering. Three years ago I visited virtually every college and university in the state in an attempt to study their major problems. Without exception I heard charges of inadequacy on the part of high school graduates; Commissioner Spaulding made an analysis of these charges in 1948, and he summarized his findings as follows:

> All too often, we are told: "High school students today can't spell." "High school grad-

uates can't use arithmetic." "High school pupils don't know the facts of American history." Or, to sum it all up: "High school students need to get back to learning the three R's as well as their grandfathers did."

Modern education has its shortcomings, without question. But the plain fact is that it is better education than the education your fathers and grandfathers had.

This is a very interesting and important conclusion, and is quite contrary to opinions of many college educators, but I discover that Dr. Spaulding had strong evidence to back him up. For example, he studied the showing of New York State high school students on the 1947 Regents tests as compared with that on the 1915 tests, and he came to the conclusion that "the present-day pupils do better in every field." About 1920, in Boston, there was discovered a complete set of examination papers, including teachers' corrections, that had been given in the schools of Boston in 1854. Boston students were given these tests, and the papers were graded in the same manner as previously. It was concluded in this study that "the pupils of this century knew the meanings of words better than did the pupils of the last century. And they could spell and figure better, too."

I have made a cursory study of the preparation of mathematics teachers now as compared with that of, say, thirty years ago. I embarked upon the analysis with a certain amount of prejudice, for I am extremely dissatisfied with the curricula now made available to our prospective secondary teachers; yet I am forced to confess that teachers now are better trained than they were thirty years ago. I may say in passing that the training of secondary teachers is becoming more and more a responsibility of our liberal arts colleges rather than the teachers colleges. In other words, there is an increasing amount of merit in the assertion that deficiencies on the part of high school teachers is a fault of the liberal arts college departments that train those teachers. It is becoming too common to blame departments of education; I have been unable to see that very many departments of mathematics are assuming any responsibility in this area. I should like to see such organizations as the American Society for Engineering Education work with institutional departments of mathematics to the end that we do a better job of training teachers of the basic courses. I have given some time to the study of certification requirements for teachers; there is little doubt in my mind, without going into an elaboration of the subject, that there is much more to the problem of obtaining good teachers than merely that of changing certification requirements. The state of New York is required by statute to maintain a program of approving, or registering, curricula for the training of those who enter the professions; we do that in engineering, medicine, and so on. Three years ago I invoked the same policy for those institutions that are training teachers; I am becoming convinced that this is a more effective device than certification. I hope the plan is extended in this country.

While I am on the subject of teaching, let me call your attention to the fact that there is a screening process going on in our colleges and universities, aided and abetted by industry, that skims off able mathematicians and scientists for research and industry whereas those left over go into teaching. This fact was revealed in striking manner by the results of the tests administered by the federal government to determine Selective Service deferments. It is my personal belief that this problem is sufficiently serious that industry can well afford to help resolve the issue. It is conceivable that science and mathematics teachers in many communities might actually become affiliates of local industries in order that they might receive added remuneration and additional experience and training. Such a program could provide strong inducements for able students to enter teaching.

I shall not dwell at length on the problems of instruction on the college level.

You know them as well as I do. However, I must confess my continual irritation that such a low caliber of teaching is tolerated on the freshman and sophomore levels in our colleges and universities. Frequently young instructors are more concerned about their research than their teaching; actual visitation to their classrooms, such as I have indulged in frequently in recent years, reveals little interest in the problems of their students. I happen to believe in good mathematics, but in some classrooms the attempts that are being made in the direction of greater rigor completely ignore the student's level of maturity and his ability to comprehend. After all, there is no such thing as absolute rigor; it is a function of a person's maturity and past experience. I think that engineers have a right to insist on better teaching from those instructors who handle the basic subject-matter fields.

So far I have indicated some of the good and some of the bad in our high school-college educational effort, as well as some pertinent factors for the development of the subject of this paper. What are some of my conclusions as I begin to synthesize these factors? First, engineering schools are observing what they regard as a deficiency in mathematical preparation on the part of high school students as compared to years past because, I believe as a partial explanation, the student of modern engineering requires much more mathematics than was required thirty years ago. A short time ago I looked over a textbook on physical chemistry that I studied nearly thirty years ago. This course was taken in the latter years of my college career, but, it seems impossible to believe, there is not a single integral in the entire book. In my total undergraduate program of science and engineering I have no recollection of using any mathematics other than the simplest algebra and trigonometry; I can still remember my surprise as a graduate student when I found some practical uses for the definite integral. The change that has taken place is little short of amazing. Now, I believe it is urgent that students actually have some calculus before they enter college if they are going to undertake the study of engineering and the physical sciences. In other words, even if tests reveal that our total educational program in high school is getting better, the college demands in such a field as mathematics have moved ahead at such a rapid pace that they have still completely outstripped what is now available in most high schools.

This leads easily into my second point. Recently I have given some attention to the total mathematics program offered by high school and college. I have become convinced of the existence of a tremendous gap between high school mathematics and college mathematics. Both in approach and content the mathematics taught on the college level and demanded on the college level is moving away from that taught on the high school level. If I had more time I could provide many specific instances of what I mean. I am advocating that the total curriculum in mathematics from the eighth grade in the schools to the end of the senior year in college must be reviewed. There is doubt that this can be done by volunteers, for it is an assignment that requires full-time study on the part of experts for a period of time. Perhaps foundation support is needed for such a project.

There is still another element in the total picture that I want to call to your attention. Engineering colleges have been increasing the amount of general education in the engineering program. That is, the amount of technical study has been diminished to a certain extent to make room for the general education that we all regard as important. However, it is interesting to observe that essentially the same thing has been happening on the high school level. I believe that I see considerable duplication of effort and a complete lack of synthesis in the field of general

Continued on page 202

The concept of a literal number symbol

MAX BEBERMAN, *Associate Professor of Education, Florida State University, Tallahassee, Florida, and*
BRUCE E. MESERVE, *Associate Professor of Mathematics, State Teachers College, Montclair, New Jersey.*

The time has certainly come to consider those experiences which will develop the concept of variable and function. It will not do to rely on words.

THIS PAPER IS CONCERNED with a concept, an idea—a part of the theory from which our mathematical procedures and rules are derived. The concept of a literal number symbol is one of the keystones of high school algebra. An understanding of this concept can bring meaning and significance to many other concepts of algebra, to applications of algebraic concepts, and to many of the manipulative skills of algebra. Thus any teacher who wants to make algebra meaningful to his students must have a clear concept of a literal number symbol. Even though meanings are enriched by experiences with practical applications of concepts, the teacher's understanding, and the student's understanding, are seriously impoverished without some insights into the theory.

In this paper we are suggesting a procedure for developing the concept of a literal number symbol. In a sense we use the procedure to emphasize the theory underlying the concept. Then we show how a graphical device—the number line—can be used to relate the concept of a literal number symbol to the concepts of equations, inequalities, and other statements involving numbers. We do not expect students to solve difficult equations and inequalities as an immediate result of the experiences described herein. The purpose of such experiences is to give the students a clear, basic understanding of equations and literal number symbols. Manipulative skill and still deeper understanding will come from additional experiences and applications.[1]

SETS OF NUMBERS

The notion of a set of objects is a familiar one to students. We have all used such expressions as "a set of books," "a set of chessmen," and "a set of silverware." Similarly, we may talk about "a set of numbers," "the set of all positive integers," or "the set of points on the number line." Each object in a set is called a *member* or an *element* of the set. Thus a spoon is a member of a set of silverware; seventeen is a member of the set of all positive integers.

Think of a set of silverware consisting of a fork, a knife, and two spoons. A statement about "a piece of silverware" may refer to each and every member of the set or to only a part of the set. For example, we may say

(1) A piece of silverware should be cleaned promptly after it is used, and

[1] The ideas and procedures in this article are based upon the experiences of the authors as members (1951–54) of the University of Illinois Committee on Secondary School Mathematics (UICSSM). The cooperation of the other members of the committee (Messrs. Babb, Nichols, Page, Pingry, and Stegmeir) in these experiences is hereby acknowledged. The point of view of the committee is evidenced in the recent multilithed revision of its ninth-year textbook entitled *High School Mathematics—First Course* (Urbana, Illinois: The Committee, University High School, 1954).

(2) A piece of silverware should be used in eating soup.

We can assume that the first statement is true for each and every member of the set, whereas the second statement refers to one of the spoons. We may think of the two spoons as a *subset* of the original set of silverware. Since the original set consists of more than the spoons, we call the set of spoons a *proper subset.*

Now consider the following statements in regard to the set of all integers:

(1) The product of an integer and zero is zero, and

(2) The square of an integer is less than twenty-five.

Again, the first statement is true for each and every integer while the second statement is true for each and every member of a proper subset of the set of all integers. For our purposes the important aspects of each of the above four statements are that an assertion is made about a particular member of a set without specifying the member, and that this assertion may be true for all members of the set or for some members of the set.

Literal number symbols

The process of making an assertion about an unspecified member of a set is facilitated by the use of appropriate symbols. We use symbols such as "7," "−3.5," and "65%" to stand for *specified* members of sets of numbers. Letter symbols such as "*K*," "*x*," and "*p*" can be used to represent *non-specified* members of sets of numbers. When letter symbols are used in this manner, they are called *literal number symbols.*[2] Numbers like 5 and $3\frac{1}{2}$ are often called *specific numbers;* numbers like x and K are often called *literal numbers.*

[2] Letter symbols can also be used to represent specified numbers. For example, the letters "π," "e," and "i" are symbols for specified numbers just as are the symbols "9" and "$\sqrt{2}$." By convention such letters are considered as symbols for specified numbers rather than as literal number symbols.

If x, y, and z are members of the set of real numbers, we may consider statements such as

$$x+y=y+x, \qquad (xy)z=x(yz),$$
$$x+1>7, \qquad xy=yz,$$
$$x+1=x, \qquad x+y=x+y+1.$$

Note that some of the above statements are true for every member of the set of real numbers and that other statements are true for every member of a proper subset of the given set. In particular, some statements are not true for any member of the given set (that is, some statements are true for only the empty or *null* subset). For example, the statement "$x+y=y+x$" is true for any real numbers x and y; the statement "$x+1>7$" is true for the subset composed of all real numbers greater than 6; and the statement "$x+1=x$" does not hold for any real number x.

Students can be asked to volunteer statements such as the above and to make judgments regarding their validity with reference to the given set. Such judgments can be made only on an intuitive basis. Students are not disturbed by the use of intuition and teachers can hardly expect more at this stage of maturity. The important consideration here is that the student is undergoing a variety of experiences with literal number symbols. He comes to recognize the literal number symbol as representing an unspecified member of a set. Students can also be asked to "translate" verbal expressions into "mathematical" expressions by using literal number symbols. This type of exercise is very common in elementary algebra classes. We suggest that such exercises also include "identities," inequalities, and other statements about numbers which are false for all members of the given set. For example, at the ninth-year level the statement

"the square of a number is equal to the square of the negative of the number"

holds for all numbers (is an identity) whereas the statement

"the square of a number is seven less than five"

does not hold for any (real) number.

Background for the function concept

Another appropriate kind of preliminary exercise provides excellent background experience for more systematic and detailed work with the function concept. Several illustrations will indicate this potential.

(1) If x may be any member of the set [1, 2, 3, 4, 5], then is it true that the sum $x+2$ may be any member of the set [3, 4, 5, 6, 7]?

(2) If the product $2x$ may be any member of the set of even numbers, then of what set can x be any member?

(3) If x and y may be any members of the set of numbers $[0, 1, 2, 3, \cdots]$, can the sum $x+y$ be any member of the same set? Can the product $x \times y$ be any member of the same set? Can the difference $x-y$ be any member of the same set? Can the quotient $x \div y$ be any member of the same set?

The concepts of domain of a variable and range of a function are readily developed by means of exercises like the above. (However, nothing of great value appears to be gained by using terms like "domain," "range," "variable," and "function" at the early stage at which the above concepts are first introduced.) Illustration (3) is laden with possibilities regarding the closure concept.

Bright students are stimulated by the implications of questions such as the following:

(4) If x may be any member of the set [1, 2, 3], then the product $2x$ can be any member of the set [2, 4, 6]. The sets [1, 2, 3] and [2, 4, 6] contain the same number of members. Similarly, if x may be any member of the set $[1, 2, 3, \cdots]$, then the product $2x$ can be any member of the set $[2, 4, 6, \cdots]$. Do the sets $[1, 2, 3, \cdots]$ and $[2, 4, 6, \cdots]$ contain the same number of members?

(5) If x may be any member of the set $[1, 2, 3, \cdots]$, how many members are there in the set of which the difference $x-x$ can be any member?

(6) If x may be any member of the set $[1, -1, 2, -2, 3, -3]$, how many members are there in the set of which x^2 can be any member? What is the relation between the numbers of members in the sets? Does this relation hold when x may be any member of the set $[1, -1, 2, -2, 3, -3, 4, -4, 5, -5]$? Of the set $[1, -1, 2, -2, 3, -3, \cdots]$?

Imaginative students enjoy speculating about the concepts of infinite set and of multivalued function involved in the above illustrations. Again, an insistence upon technical terminology serves no constructive purpose at this stage of development; indeed, it may have a negative effect.

The number line

Statements about numbers can be treated in an interesting manner when the statements are interpreted geometrically. The number line (Fig. 1) is an excellent visual device for this purpose. In the discussion which follows we assume that students are familiar with signed numbers and zero, with the symbols

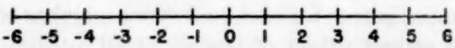


Figure 1

$$>, =, <, \geqq, \leqq, \neq,$$

and with the symbol for "absolute value."[3]

Students readily accept on an intuitive basis the idea of the one-to-one correspondence between the set of real numbers and the set of all points on the number line. They agree that each member of the set of real numbers is the *coordinate* of a unique point on the number line, and that each point on the number line is the *graph* of a unique real number. Any specified real number is the coordinate of a particular (specified) point on the line. If a number x may be any real number, then the point with coordinate x may be any point on the number line.

The student is now ready to use the number line in discussing geometric inter-

[3] The number line is studied in the first unit of the ninth-year textbook (*op. cit.*) used at the University of Illinois High School. The procedures described in the present section can be adapted for use with students who have not studied signed numbers. The symbols may be easily defined when needed.

pretations of statements involving literal number symbols. The discussion is facilitated by assigning letter names as well as coordinates to some of the points. (See Fig. 2. Students recognize that only a portion of the number line can be depicted

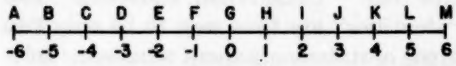


Figure 2

and that the line extends indefinitely in both directions.) Given any statement, the student is to describe a set of points (a "locus") with the following characteristics: (a) each member of the set has a coordinate which makes the given statement a true statement, and (b) the set contains all such points. Intuition, again, is the student's primary arbiter in judging the acceptability of his responses. In effect, the student is asked to describe sets of real numbers (the total set, proper subsets, or the null set) for which given statements are true. What purpose does the number line serve? We can best illustrate the value of the number line by describing a teaching situation involving a typical exercise.

Exercise. Describe the set of points with coordinate x such that $x^2>16$.

Suppose a student responds, "These are the points on the right of K." (See Fig. 2.) The teacher may indicate this region on the number line by appropriate shading or cross-hatching and proceed somewhat as follows.

"Now we must verify this statement. Let's select a point from this set, say point M. Does its coordinate make '$x^2>16$' a true statement? It does. Try another point. . . . Do you think every point in this set will work? . . . (Yes.) Now, do we have *all* the points which will work? Let's try some points which are not in Bill's set. Try I. . . . (No, its coordinate doesn't make '$x^2>16$' a true statement.) Try D. . . . (No.) Try B. . . . (Ah, B works.) Try A. . . . (It works, too.) Do you think that all points on the left of B will work? . . . (Yes.) Now try C. . . . (It doesn't work.) How about a point between C and B, say the one whose coordinate is $-4\frac{1}{2}$? . . . (It does work.) Now then, who can describe the set containing all the points, and only those points, whose coordinates make '$x^2>16$' a true statement? . . . (All the points on the right of K and all the points on the left of C.) Do you see that Bill's set contained *some* of the points but not *all* of them?"

The number line serves the same purpose for the student in elementary algebra as number scales serve for children in grade school. We have a graphic device which gives us a picture of the real numbers *in order*. Such a picture permits a systematic investigation and verification of statements involving real numbers. Most students need the assistance of this concrete device.

The following statements indicate the great variety of exercises that may be considered using this approach.

1. $x+1=3$	2. $x>2$
3. $x<-3$	4. $x-1<3$
5. $x-1>3$	6. $1-x>3$
7. $\|x\|=4$	8. $\|x\|>4$
9. $\|x\|<4$	10. $x+1=1+x$
11. $x+x=2x$	12. $\|x\|>0$
13. $\|x\|<0$	14. $x\geqq 2$
15. $x\neq 3$	16. $x^2>25$
17. $(x-2)^2=0$	18. $(x-2)^2>0$
19. $\|x-2\|>0$	20. $x=x$
21. $x\div x=1$	22. $x>x+1$
23. $x<x+1$	24. $5<x<6$
25. $5<x>6$	26. $5>x<6$
27. $5>x>6$	28. $1<\|x\|<2$
29. $2x$ is an even number	30. $2x$ is an odd number
31. x is not an integer	32. $3x$ is an integer
33. $6x$ is not an integer	34. $x=\|x\|$
35. $x+2=\|x+2\|$	36. $\|x+2\|=\|x\|+2$
37. $\|x+2\|\leqq\|x\|+2$	38. $\|2x\|=2\|x\|$
39. $\|-2x\|=2\|x\|$	40. $x^2=-4$
41. $(x+1)^2=x^2+1$	42. $\sqrt{x^2+4}=x+2$

The converse type exercise is also interesting. For example, the student might be asked to make a statement that is true only for the integers 1 and -1, or to make a statement that is true only for the real

numbers between -2 and 2. Such exercises encourage the student to discover that answers are not unique and thus to develop the idea of equivalent statements.

The reader will recognize that the possibilities for mere mechanical performance in the above exercises are exceedingly remote. Certainly this is a desirable feature. Students usually spend a short period of time in informal work developing meanings for a concept and then build mechanical skill based on the concept. Too often the period of concept development is too short to permit the acquisition of basic meanings. Since mechanical performances do not require constant recourse to meanings, the learners need an assimilation period in which they can spend sufficient time with a concept and in which they are provided with practice consciously designed to bring meaning to the concept.

Conclusion

The purpose of this paper was twofold. We endeavored to present a theoretical description of the role of the literal number symbol. We defined a literal number symbol as a letter which represents an unspecified member of a set of numbers. We demonstrated the usefulness of this symbol in making statements about numbers, and indicated that its usefulness is based upon its non-specifying character. Thus a statement can be made without "immediate" knowledge of its truth. To say that a statement involving a non-specified number is true means that you have found a subset of the real numbers such that when each member of this subset replaces the non-specified number in the statement, the resulting assertion is a true one. "Conditional equations" and "identities" are thus seen to be only two kinds of statements about numbers; "solving an equation" is a process of finding the desired subset. We feel that all teachers of algebra need to have a theoretical understanding of the role of the literal number symbol in order to be able to provide those experiences which will result in a meaningful concept for their students.

Our second purpose was to illustrate the kinds of experience that help students acquire understanding. Students will develop a meaningful concept of the literal number symbol as a symbol for an unspecified member of a clearly defined set by actually constructing such sets as in the "function concept" illustrations. Students will increase their understanding of the concept of an unspecified number (and develop a sound concept of the equation) by considering a great variety of number statements and determining the conditions under which each is true. Students will also gain understanding of these concepts through experiences with formulas, the number line, and other applications.

Some educational problems

Continued from page 197

education as it is handled upon the high school level and then on the college level. I strongly suspect, and this may be heresy, that a complete review of the total high school-college program might reveal some opportunity for now reversing to a certain extent some of the trends of the last two decades and thereby introducing some additional technical study. This comment emphasizes again that our problem is essentially one of high school-college articulation. The problems must be faced objectively in conferences by educators from both high school and college.

Take a number

L. M. WEINER, *DePaul University, Chicago, Illinois.*

A little spice to whet the appetite of the algebra class.

IT IS THE PURPOSE of this article to call to your attention a device which is quite old but one which I have nevertheless found very useful in teaching how to simplify an algebraic expression and how to solve a linear equation. The method consists of a mind-reading trick. The instructor asks the members of the class to choose a number. "Now," he says, "In most mind-reading tricks, the performer tells you what to do to your number. I am going to let you do whatever you want to your number. However, you must tell me what it is you are doing."

He then proceeds to call upon members of the class to suggest arithmetical operations for them to apply to their number. After several people have made suggestions, he says, "Let me get in on this also." The instructor then dictates several more operations for the class to perform and then announces to them what their number is.

The most instructive part of this comes in explaining to the class how the final number is determined. Let us see how this is done. When the class first chooses a number, the instructor is of course in the dark as to what this number is. He therefore employs one of the fundamental concepts of algebra. "When we wish to perform calculations with a numerical quantity whose value is unknown to us, we write down the symbol x for this quantity and proceed to work with it as if its value were known."

Having written down the symbol x, he then proceeds to apply to it the operations suggested by the members of the class. For example, if the operations mentioned are add 4, divide by 2, multiply by 3, add 6, divide by 4, and subtract 5, the expressions written down are x, $x+4$, $(x+4)/2$, $(3x+12)/2$, $(3x+24)/2$, $(3x+24)/8$, and $(3x-16)/8$ respectively. At this point the instructor takes over. His purpose is to simplify the final expression to the form ax. He says therefore, "Multiply by 8 and add 16." This leaves $3x$, so his final instructions are, "Divide by the number you started with," and he then knows that each member of the class should have the result 3 at this point, provided he carried out the computations correctly.

Aside from the recreational aspects of this problem, it seems to give the student a certain insight into the need for simplifying algebraic expressions. After all, an equation is nothing more than two algebraic expressions separated by a sign of equality. The objective is to simplify the equation, if possible, to the form $x=a$, keeping in mind of course that an equation is analogous to a scale in balance in that whatever one does to one side, he must do also to the other side if he wishes to maintain the balance. Here, the student is working only on one arm of the scale, so to speak, whereas, in solving equations he must keep both sides in balance.

The students are usually anxious to try this trick on their friends, and in so doing, they are forced to manipulate the algebraic expressions involved. Thus the problem provides some pleasant homework for them. There are many variations of this scheme which may be utilized in the classroom, and with a bit of careful planning, the individual teacher may devise these for himself.

Mathematics and general education

LYLE J. DIXON, *Arkansas State College, State College, Arkansas.*

What criteria should be considered in developing a mathematics course for general education purposes? The author lists a few and discusses types of general education courses.

DURING THE PAST DECADE we have witnessed in our colleges and universities a changing program of general education. Many of us have been directly affected by this "new" educational philosophy. In discussions and correspondence with fellow mathematics teachers one learns that much time is, and has been, spent in discussions of this topic at faculty meetings, organizational meetings, and informal sessions over coffee. The literature contains many references to general education, its processes, its advantages, its weaknesses, evaluation of programs of general education, and many of the other aspects of this educational movement. One thing is noted in all of these various references; that is, there are almost as many meanings of general education as there are individuals who are concerned with it. There are many definitions. However, a broad description of general education includes the emphasis on a common body of knowledge which is necessary for effective living in our present society.

Two things seem to stand out in most programs of general education. First, the program is done up in "package form." The form ranges from a distributional pattern, with courses selected by the student from various specified areas, to a rigid requirement of specific courses for graduation. The form is, and should be, the choice of the individual college and university. In some cases it is noted that the selection of the form influences the effectiveness of the program. It might also be noted that most colleges and universities which have a program of some kind have the major portion of their work in general education courses in the freshman and sophomore years. Some schools devote the first two years entirely to general education. It appears that a program might be more successful if it were distributed over the full four years with a little more emphasis on general education during the first two years than in the last two years. This permits a student to do a little investigating in areas in which he has interest and allows time for some specialization early in the college career.

Second, the programs seem to be weighted in favor of the social sciences. This may be due, in part, to the emphasis upon human needs. Educators are inclined to feel that human needs are far more important than in the other areas. Emphasis on ability to communicate effectively, knowledge of social institutions and their functions, acquaintance with art, music and literature, and, to a certain extent, understanding of economic conditions overshadows any program in the sciences and in mathematics.

Those institutions which have respectable programs in science often relegate mathematics to the cellar position of a tool. All of us are aware of this fact and are inclined to criticize others for letting this situation exist. How much of this is our own fault and not the fault of someone else? We do not have an inferior product, educationally speaking; we are just poor salesmen. Too few in the mathematics profession have taken the time to

examine fully and completely the role of mathematics in general education. We have, so to speak, been too busy complaining (mostly privately) to assume any leadership and offer constructive programs in mathematics and science.

Perhaps these are harsh words to put before the profession of mathematics teachers, but I firmly believe there is ample evidence to warrant these words. E. A. Cameron[1] made the statement that "In institutions which have inaugurated 'general education' courses in the humanities, the social sciences, or the natural sciences, with but a single exception mathematics has no place in any of these courses. Indeed, in some institutions which have had the longest experience with such courses the mathematical needs of only those students who follow certain special curricula are seriously considered. *This suggests that some mathematics departments are not especially concerned with the contribution their subject might make to a liberal education or else feel that this contribution is automatically provided by the usual introductory courses*" (not italicized in the original).

If we believe in mathematics and what it has to offer, we must strive to balance the general education program and include in the pattern of courses a type of mathematics course which will make the best possible mathematical contribution in developing intelligent college students.

After one becomes aware of the need for such a course in mathematics for the general education program, the question arises, "What is it we would like to accomplish in this new course?" In other words, what are the objectives? One becomes cautious at the mention of the word, objectives. These are often statements formulated to provide a reason for what one is doing. Besides, the journals are full of many statements regarding objectives for courses in mathematics for general education. One should note that any objectives which might be thought desirable must be in harmony with and supplement the objectives of the individual institution in which the course is being developed.

I do not intend to offer a new formulation on objectives. My experience has induced me to believe that the fundamental objective of such a course is the development of mathematics as a way of thinking. This seems to be in harmony with most lists of objectives that may have been formulated. If one attempts to do this in a course, several "by-products" are, of necessity, produced. These are often considered as the *intermediate objectives.* Among these "by-products" one will find an understanding of the nature and significance of mathematics. Fundamental processes can be developed along the way. Skills and techniques become a means to achieve the objective and not the objective itself. Logical processes receive proper attention and the relation of mathematics to all other areas of human endeavor becomes a must.

This statement of belief may be logically equivalent to other more extensive formulations of objectives. Many teachers probably have lists of objectives which indicate the direction they would like to go much better than this brief statement.

We could list objectives by the hour and accomplish only one or two things. We would have an idea of where we would like to go, educationally speaking, if an argument did not destroy our common efforts on this question. What is more important, and this is where each of us has his or her own ideas, is how we go about achieving these objectives.

Several factors seem to influence what we choose in the way of content. Besides the local and geographical aspects which are ever present about an institution, the type of student enrolled in the college or university will influence the approach any given department might take. Other related factors must also be taken into

[1] E. A. Cameron, "Some Observations on Undergraduate Mathematics in American Colleges and Universities," *American Mathematical Monthly*, Vol. 60 (1953), pp. 151–55.

account in the development of any course. What is more important in this discussion are the factors which bear directly on the question of mathematics and general education. There is a growing feeling of dissatisfaction in the traditional program of mathematics and its inadequate contributions to general education. There are many indications that this feeling is justified, but teachers, and it seems particularly true of mathematics teachers, are reluctant to change.

Courses should be continually revised and improved. A recent book advertisement that came to my desk emphasized the lack of change in the fourth edition as compared with the first. This kind of philosophy is not conducive to an evolving program of mathematics and is not conducive to the development of the type of mathematics needed for a general education program.

One should also be aware that a mathematics course designed for general education must have *real* mathematics as its core. Courses in which one talks about mathematics do not insure that the desired objectives will be achieved. It has been noted many times that one does not learn what mathematics is like by talking about it. On the other hand, too much emphasis on skills and techniques will also destroy the effectiveness of a course in mathematics. Techniques are necessary, but they must be in the right proportion.

Individual factors within a given college or university will also influence the kind of content within a given course. This individuality is often the source of many over-all improvements in the teaching of mathematics.

One finds a variety of mathematics courses for a general education program. "Variety is the spice of life" seems to apply here. "There is more than one way to skin a cat" also seems to apply. The different approaches that have been used to achieve the objectives show that it is possible for programs to differ greatly in terms of content and yet equally satisfy the job of filling a role in general education. It is possible to observe at least five types of courses which are, or have been, used as a part of a general education program.

First, traditional courses of algebra, trigonometry and, sometimes, analytics and calculus are open to students in the mathematics area. It seems that traditional courses are used more in those colleges and universities which have a distributional plan where the student must select some course in a given area. Serious questions have been asked concerning the effectiveness of this type of approach. Can a course, say in college algebra or trigonometry, be both a technical preparation for future work and an *effective* contribution to general education? There have been many arguments on this question and it can be fairly certain that the debate is not over.

Some colleges and universities have revised their traditional programs and now have a series of unified courses. This second type of course contains much the same material, but it is presented in a manner which will unify and build mathematics as it really is. I have yet to hear a complaint about the effectiveness of such courses. It is a case of improved teaching methods and curriculum construction. The effectiveness of the unified courses from the general education standpoint surpasses the effectiveness of the traditional courses. Yet we seem to be too busy developing techniques to place much emphasis upon what mathematics is and the philosophy behind mathematics. The success of the unified course depends upon the teacher and his ability to unify the many so-called branches of mathematics.

A third type is often referred to as a "cultural" course in mathematics. This is the general description of what might occur when one talks *about* mathematics. This type does not appear to be too successful and there is a strong belief in many quarters that it is not even mathematics. It seems that the central theme of such courses is the development of mathematics

and its relation to the development of civilization; as a history of mathematics it often surpasses the traditional course in its historical evolution. Many schools have tried this sort of approach and have dropped it. The main reason seems to be the lack of success with the course. Whether this failure is caused by a lack of sympathetic instructors, or other factors, one can never be sure.

A fourth approach to the question is found in what might be called "technique" courses. These courses emphasize the practical aspects of mathematics, such as business mathematics, agriculture mathematics, vocational mathematics, etc. Mathematics itself is spread pretty thin. It appears that these courses have missed the main objective; that is, that mathematics is a way of thinking and not simply a tool. The "tool" aspect is important, for we want people to use mathematics. It has much to offer. But to say that mathematics is merely a tool is a gross injustice against mathematics and against every student. Unfortunately the emphasis in secondary schools has, in the immediate past, been in this direction and some schools still insist on this emphasis.

A passing note might be made that types 2, 3, and 4 often appear under the broad title of general mathematics (a term soundly over-worked).

E. A. Cameron[2] gives a brief description of the fifth approach when he says, "The type of course which offers most promise of substantial contribution to a general education is not adequately described by mere listing of topics covered. The spirit in which the subject is treated is of the greatest importance. An understanding of the nature and significance of mathematics is sought through an emphasis on the basic concepts, the logical processes used in developing the subject, and the relation of the discipline to other fields through a consideration of its origins and its applications." This approach seems to be a presentation of mathematics as it really is and has a strong core of mathematics for its center. It is, in most cases, quite unlike traditional courses. The approach is still in its infancy. Many aspects have yet to be considered and the effectiveness of this type of course has not as yet been measured. Those teachers who have taught such a course seem convinced it has immense possibilities, more so than any other attempt at teaching mathematics for general education.

Unfortunately, there are too few texts available which can be used for this type of course, and it is difficult to prepare material which can be used. The instructors are usually burdened with a teaching load and other duties which prevent a concentrated attempt at preparing outlines and material for such courses. Many of our mathematical leaders who could make valuable contributions are too busy with research and other activities to devote time to such things as this.

The thing which seems to make this last approach so successful is the "newness" of the approach itself. Those institutions which have had success with the courses have used staff members who possess a philosophy which is in harmony with the purposes of general education. The course outlines themselves reflect something "new," something invigorating. For example, an outline of a course at a certain midwestern college shows that the content is organized completely around the idea of logic. Another institution has used mathematics as a language, as the central theme. It is not uncommon to see discussions of groups, algebra of logic, and other topics from modern mathematics as topics of discussion in these courses. The writer has achieved remarkable success with a course outlined around the concept of sets.

From personal contacts it is easy to see that many are attempting to devise something which is along the pattern of this fifth approach.

In conclusion, it might be well to note

[2] *Ibid.*, p. 154.

some fundamental principles in the construction of a mathematics course for general education.

The course:

1. Must be mathematical in character, not simply a discussion of what mathematics is about;

2. Must be based upon the conditions which are found within the given school (it is almost impossible to borrow a course from another institution and achieve success with it);

3. Must be based upon sound objectives, objectives which are the outgrowth of a serious study of what the over-all program should achieve;

4. Must be developed by the people who are going to teach it; and

5. Must be a development of mathematics for a modern world. In this sense, the use of such things as mathematics as a language, logic, sets, functions, etc., as fundamental themes has resulted in courses that have been excellent examples of a modern mathematics.

It seems almost certain that a course constructed upon these fundamental principles will survive and provide an approach whereby the typical college student can gain an insight into and a partial understanding of the vast realm of mathematics.

A note on teaching percentage

Percentage continues to be one of the main trouble spots in arithmetic, whether it be in the junior high school or in courses in shop mathematics or business arithmetic in the senior high school or college. There is overwhelming evidence to indicate that by and large, students do not attain assured mastery of percentage. In view of its extreme importance as an instrument for thinking about and dealing with commonplace matters, however, it is important that they do so. As teachers we can hardly be complacent about the situation as it is now.

There seem to be several things that contribute to the confusion that is so prevalent with respect to percentage. The terminology that has come to be universally employed is itself confusing and unfortunate. In the identification of "per cent" with "hundredths" the base number often is not mentioned, though neither of these concepts has meaning except as referred to some base. The further representation of, say, "37 per cent" or "37%" by the decimal fraction "0.37" fails to emphasize the clarifying idea that 0.37 itself means 37 (0.01) and that the % symbol merely stands for the decimal fraction 0.01. These alternative representations of the same thing thus tend to add to the student's confusion, especially when such expressions as "0.37%" are encountered. Experienced teachers will verify that students often fail to make any distinction in their minds between the expressions "37%" and "0.37%." When the subject is taught by "cases," three explicit rules can indeed be developed for the three explicit relations among the three elements involved, but these tend to become confused in the students' minds, and students often have trouble determining "which case this is." Even the single over-all rule or formula *base* × *rate* = *percentage* (or $b \cdot r = p$) requires that the rate must be represented by a decimal fraction only, and not by use of the per cent sign, yet even this is seldom emphasized in the textbooks.

These points that have been mentioned are simple but crucial, in that they represent foci of serious difficulties. Perhaps by focusing students' attention upon them and by explaining and illustrating them with patience and care, teachers may be able to bring their students to clearer understanding of percentage and its applications.—*Charles H. Butler, Kalamazoo, Michigan*

"All the high hopes which I entertain of a more glorious future for the human race are built upon the elevation of the teacher's profession and the enlargement of the teacher's usefulness."—*Horace Mann*

Seminars—an integrating force in a program of concentration

SISTER MARY CORONA, O.S.F., *Cardinal Stritch College, Milwaukee, Wisconsin.*

In these days of changing mathematics courses and mathematical curriculums, it is good to see a description of an undergraduate program for mathematics majors.

IN 1946 the Curriculum Committee of Cardinal Stritch College inaugurated a pattern differing from the usual major-minor program of specialization adhered to in the past decade. A *Plan of Concentration*, similar to the form worked out at Princeton and Cambridge and gaining considerable recognition in colleges of the east, was adopted as the pattern for the Bachelor of Arts degree in the biology, English, history, and Latin departments.

In a College Curriculum Workshop conducted at the Catholic University of America in 1947, a sequence of courses for these departments was established. Other phases of the program of concentration were worked out at the Workshop of College Administration and Organization, Catholic University of America, in 1947, 1949, 1951 and 1952.

Prior to 1950, the college offered a minimum of eighteen credits in the department of mathematics. In 1950 the department was requested to develop a full program of courses on a pattern of concentration.

The first years under this program, though highly experimental, soon proved the concentration program a worthy replacement of the old pattern. This program equipped the student with a broader knowledge and a fuller understanding of his field of concentration.

The pattern of concentration as followed at the college (Table I) consists of Lower and Upper Division courses, each charged with specific objectives. The Lower Division courses offer a program of *general education in the liberal arts.* After the completion of these courses, the student should emerge with an adequate control of the means of communication, a philosophy of life, and an academic preparation necessary for concentration in a special field. Since students in the Lower Division have much in common with high school students, the educational experience of the first two years of college is planned to be more akin to the program of the secondary school.[1] At present, all B.A. curriculums are organized according to this plan in the first two years while the last two years, or Upper Division, are more largely devoted to specialized fields of concentration.

In addition to the general aims of Cardinal Stritch College and the objectives to be achieved by the concentration program, the department of mathematics includes three particular aims:

a. To furnish the students with mathematical tools as needed in the field of physics, chemistry, and statistical sciences offered in the steps of general education.

b. To offer the students a program of

[1] "Self-Evaluation Report," Cardinal Stritch College, Milwaukee, Wisconsin, p. 15.

TABLE I

		LOWER DIVISION					
		Freshman Year					
		First Semester				*Second Semester*	
Th	101.	The Supernatural Man	1	Th	102.	The Supernatural Man	1
En	101.	Composition and Lit.	3	En	102.	Composition and Lit.	3
Hs	101.	European Civilization	3	Hs	102.	European Civilization	3
______	.	Chemistry or Biology	4	______	.	Chemistry or Biology	4
Mt	101.	College Mathematics	3	Mt	102.	College Mathematics	3
______	.	French or German	3	______	.	French or German	3
PE	.	Elective	0	PE	102.	First Aid	0
			17				17
		Sophomore Year					
Th	201.	The Supernatural Man	1	Th	202.	The Supernatural Man	1
En	201.	Survey of Western Lit.	3	En	202.	Survey of Western Lit.	3
Pl	201.	Rational Psychology	3	Pl	202.	Logic	3
Py	201.	General Physics	4	Py	202.	General Physics	4
Mt	201.	Differential Calculus	4	Mt	202.	Integral Calculus	4
______	.	French or German	3	______	.	French or German	3
			18				18
		English Composition Examination					
		Modern Foreign Language Examination					
		Sophomore Rating Test					
		UPPER DIVISION					
		Junior Year					
Th	303.	The Supernatural Man	1	Th	304.	The Supernatural Man	1
Pl	303.	Epistemology	3	Pl	304.	Metaphysics	3
Mt	301.	Directed Reading	2	Mt	302.	Directed Reading	2
Mt	303.	Advanced Calculus	3	Mt	304.	Theory of Equations	3
Mt	305.	Statistics	3	Mt	306.	Statistics	3
______	.	Elective	3	______	.	Elective	3
			15				15
		Senior Year					
Th	403.	The Supernatural Man	1	Th	404.	The Supernatural Man	1
Pl	403.	Ethics	3	______	.	Elective	3
Mt	401.	Co-ordinating Seminar	2	Mt	402.	Co-ordinating Seminar	2
Mt	403.	College Geometry	3	Mt	404.	Differential Equations	3
______	.	Elective	3	______	.	Elective	3
______	.	Elective	3	______	.	Elective	3
			15				15
		Comprehensive Examination					
		Graduate Record Examination					

teacher-preparatory training for the elementary and secondary schools.

c. To develop in the individual the ability to do sound, accurate thinking, and to form right judgments.

With these objectives in mind, the following requirements have been set up for the Lower Division courses: college mathematics, two semesters; differential calculus and integral calculus, one semester each; and physics, two semesters. College mathematics, which includes college algebra, trigonometry, and analytical geometry, is offered in the first year. The course in college mathematics is an excellent tool for chemistry, biology, physics, and history concentrators. Differential calculus and integral calculus are offered in

the second year since they are found to be indispensable for higher courses in the physical, chemical, and statistical sciences. Physics, also a required course for concentrators, gives the universal principles upon which the necessities and conveniences of life are built. Our present scientific age demands a knowledge of such physical principles.

At the conclusion of the Lower Division courses the student is required to take the following terminal tests: the Sophomore Rating Test, the English Composition Examination, and the Modern Foreign Language Test. Remedial work is required if a specific norm is not achieved.

Concentration in the Upper Division is comprised of the following courses: advanced calculus, theory of equations, elementary and advanced mathematical statistics, college geometry, modern algebra, and differential equations. These subjects are eventually integrated in two courses offered in the junior and senior years: the Directed Reading and Co-ordinating Seminars.

The Directed Reading Seminar is the basic feature in the first year of concentration. As the title suggests, wide outside reading under direction of an advisor is recommended. The course allows for individual conferences as well as for group guidance and criticism. The main objective, as in other departments, is to train the student to read for *content*, to cultivate the habit of forming judgments intelligently and objectively, and to develop the ability to organize logically the materials taken from very broad and involved fields. It is understood that the concentration program in the Reading Seminar is not a specialization in the graduate sense. The student is merely getting an experience which is a part—the most characteristic part—of what a liberal education demands in the light of the enormous reaches of present-day knowledge. Let it be understood, too, that the student is not engaged in adding to mankind's knowledge; rather, he is engaged in acquiring more knowledge.

The subject matter of the Directed Reading Seminar comprises the history of mathematics, the fundamental concepts, the philosophy of mathematics and some study of periodical literature. In the history of mathematics, emphasis is placed on development of mathematics rather than on historical facts. Thus the present tendency of overspecialization is somewhat counteracted. The second semester is devoted to readings from current publications and a study of fundamental concepts and philosophy of mathematics.

Our senior concentrators participate in a Co-ordinating Seminar which elevates their concept of mathematics to an even higher level; namely, that of intellectual comprehension of the field as an organic whole and as a basis for scientific thinking and acting. Consequently, the last year of integral work requires a certain degree of intellectual maturity and a readiness to do independent thinking. Actual contact with the *content* of living mathematics is necessary. The Co-ordinating Seminar offers a deeper understanding of the subject matter and demands concentrated effort.

The Co-ordinating Seminar studies the structure of mathematics through the fields of algebra, geometry and analysis. The theory of numbers is used as an introductory approach. This includes such topics as Number Systems; Properties of Numbers; Euclid's Algorism; Prime Numbers; Euler's Factorization Method; Mersenne and Fermat Primes; Aliquot Parts; Properties of Congruences; Residues; Analysis of Congruences; Linear Congruences; Simultaneous Congruences; and the Chinese Remainder Theorem.

Some algebraic, theoretical and analytical topics discussed and studied are Horner's, Cardan's, Ferrari's, and Newton's Methods of Solving Equations; Envelopes; Evolutes; Involutes; Singular Solutions of Differential Equations; Determinants; Matrices; Resultants, Discriminants, and Eliminants.

The field of geometry may include Non-Euclidean Geometry, the Fifth Postulate;

Projection and Section; Ideal Elements; Principles of Duality; Simple and Complete Figures; Configuration of Desargues; Harmonic Ranges and Harmonic Pencils; Cross-Ratios and Harmonic Scales; Circle of Appollonius; Pascal and Brianchon Theorems.

Worthy of note is the fact that after following such a program the student has an over-all picture of the various fields of knowledge which bear upon his area of concentration.

A tentative reading list (see below) for both seminars was designed to aid the student in the enterprise of concentration. Honestly followed, it will prepare the individual adequately for the Comprehensive Examination, an important testing device at the termination of this program. Unless the student gives evidence of a deep and wide experience in his specialized subject with a degree of mastery that may be expected of a college senior following a liberal arts program, he will be withheld from obtaining his degree.

In the near future we hope to make a comparative study of the ratio of success exhibited between graduates of a major-minor program and those under our program of concentration.

Reading List for Concentrators in Mathematics

Reading Seminar

A. Required reading in:

Bell, E. T., *The Development of Mathematics*, McGraw-Hill, New York, 1945

Cajori, F., *A History of Mathematics*, Macmillan, New York, 1919

Courant, R. and Robbins, H., *What Is Mathematics?* Oxford University Press, New York, 1941

Young, J. W., *Fundamental Concepts of Algebra and Geometry*, Macmillan, New York, 1923

B. Supplementary reading in history:

Ball, W. W., *A Short History of Mathematics*, Macmillan, New York, 1927

Ball, W. W., *A Short Account of the History of Mathematics*, Macmillan, New York, 1922

Cajori, F., *A History of Elementary Mathematics*, Macmillan, New York, 1919

Sanford, Vera, *A Short History of Mathematics*, Houghton-Mifflin, Chicago, 1930

Smith, D. E., *History of Mathematics*, Ginn, Chicago, 1923

Smith, D. E., *History of Mathematics in America Before 1900*, Open Court, Chicago, Illinois, 1934

C. Supplementary reading in the nature of mathematics:

Dantzig, *Number, The Language of Science*, Macmillan, New York, 1939

Kerschner-Wilcox, *The Anatomy of Mathematics*, Ronald, New York, 1949

Keyser, C. J., *Mathematical Philosophy*, Dutton, New York, 1922

Maziarz, E., *The Philosophy of Mathematics*, Philosophical Library, New York, 1950

Co-ordinating Seminar

A. Several topics from each of the four branches are required:

I. *Analysis*

Hardy, G. H., *Pure Mathematics*, University Press, Cambridge, 1939 (Chapters 1 and 2)

Ore, Oystein, *Number Theory and Its History*, McGraw-Hill, New York, 1948 (Chapters 1–13)

Mathewson, *Elementary Finite Groups*, Houghton-Mifflin, Chicago, 1930 (Chapters 1, 2, 3, 4 and 5)

Townsend, E. J., *Functions of a Complex Variable*, Henry Holt, New York, 1915 (Chapters 1, 2 and 3)

Uspensky and Heaslet, *Elementary Number Theory*, McGraw-Hill, New York, 1939 (Chapters 1–9)

II. *Algebra*

Agnew, R. P., *Differential Equations*, McGraw-Hill, New York, 1942 (Chapters 1–7)

Albert, A., *Introduction to Algebraic Theories*, Chicago University Press, Chicago, 1941 (Chapters 1, 2 and 3)

Bernard, S., and Child, J., *Higher Algebra*, Macmillan, New York, 1937 (Chapters 31 and 32)

Bocher, M., *Introduction to Higher Algebra*, Macmillan, New York, 1947 (Chapters 1, 2, 3, 4 and 5)

Brand, L., *Vector and Tensor Analysis*, John Wiley, New York, 1947 (Chapter 1)

Burnside, W. and Panton, *The Theory of Equations*, Vol. 1, Hodges, Figgis, Dublin, 1928 (Chapter 11)

Dickson, *First Course in the Theory of Equations*, John Wiley, New York, 1922 (Chapters 1–9)

Taylor, J., *Vector Analysis*, Prentice-Hall, New York, 1939 (Chapter 1)

Thomas, J. M., *Theory of Equations*, McGraw-Hill, New York, 1928 (Chapters 1, 2, 3 and 4)

Weiss, Marie, *Higher Algebra*, John Wiley, New York, 1949 (Chapters 1, 2, 3, 4, 5, 6, 7 and 9)

III. *Geometry*

Dowling, L., *Projective Geometry*, McGraw-Hill, New York, 1917 (Chapters 1, 2, 3, 4, 5, 6, 7 and 8)

Johnson, R. H., *Modern Geometry*, Houghton-Mifflin, Boston, 1929 (Chapters 1–12)

Maxwell, E. H., *Geometry for Advanced Pupils*, Clarendon Press, Oxford, 1949 (Chapters I, II, III)

Veblen and Young, *Projective Geometry*, Ginn, Chicago, 1910 (Reference only)

Winger, *Projective Geometry*, Heath, Chicago, 1923 (Chapters 1–9)

Winsor, A., *Modern Higher Plane Geometry*, Christopher, Boston, 1941

IV. *Statistics and probability*

Bernard and Child, *Higher Algebra*, New York, 1936 (Chapter 32)

Coolidge, J. L., *Introduction to Mathematical Probability*, Oxford University Press, New York, 1925 (Chapters 1–3)

Mode, E., *Elements of Statistics*, Prentice-Hall, New York, 1951 (Chapters XII and XIII)

Yule, G. V., *Introduction to the Theory of Statistics*, Philadelphia, 1932 (Chapters 1, 2, 6, 7, 8, 9 and 12)

Words! words! words!

One of the most astonishing phenomena in the field of mathematical education is the immense quantity of paper that has been used writing about the function concept, and the apparent failure of instructional materials and of some teachers to make use of what has been written.

It is quite easy to find that "a variable is a quantity that varies." It is not at all surprising to find that, in the words of some books on methods, "A function is a quantity which depends upon another quantity for its value." Is it any wonder that high school pupils never find out—even faintly—how a mathematician uses the word function? These words say absolutely nothing. The concept of function is basic to the study of mathematics. However, it seems that it has not occurred to some that the above combination of meaningless words will never tell a high school student how the word function is used in mathematics.

While it is not possible to develop an "ordered number pair" definition, or even repeat the more usual "one-to-one correspondence" definition, in this limited space, it is possible to point out that psychological considerations dictate that the immature high school student see, or experience, several instances in which the function concept is brought into play. These instances should be varied—more varied than is usually the case. Examples should be presented in which the word "dependence," in its popular sense, could not be used. For example, if x and y denote an arbitrary first member and second member, respectively, of the following pairs of numbers: (2, 3), (4, 5), (5.6, 9), and (1, 1), then y is a function of x and the set of number pairs is said to exhibit a function. Here is a very simple idea which can be developed through experience, but which can never be developed by relying on words, particularly, if the words are of the kind: "quantities which depend upon one another." Words do not convey ideas if a certain background of experience is absent.

If the teacher does not develop this experiential background carefully, instruction becomes a mere conglomeration of sounds; a memorization of WORDS, WORDS, WORDS.—*Henry Van Engen*

"The name of the song is called 'Haddocks' Eyes'."

"Oh, that's the name of the song, is it?" Alice said, trying to feel interested.

"No, you don't understand," the Knight said, looking a little vexed. "That's what the name is called. The name really is, 'The Aged Aged Man'."

"Then I ought to have said 'That's what the song is called'?" Alice corrected herself.

"No, you oughtn't: that's quite another thing! The song is called 'Ways and Means': but that's only what it's called, you know!"

"Well, what is the song, then?" said Alice, who was by this time completely bewildered.

"I was coming to that," the Knight said. "The song really is 'A-sitting On a Gate': and the tune's my own invention."—*Lewis Carroll*

Mathematics—a language[1]

GEORGE R. SEIDEL, *Research Supervisor of E. I. duPont deNemours and Company, Wilmington, Delaware.* *From industry's point of view, the readers of* THE MATHEMATICS TEACHER *are told about some of the primary responsibilities of the mathematics teacher. What does it take to train a student to reason clearly?*

SINCE NOTHING IS NEW under the sun, I am certain that what I have to tell you is far from novel. I do hope to say a few things that mathematics teachers will like to hear coming from an ex-math teacher and one now engaged in industrial chemical research.

In the first place, you are members of one of the indispensable professions ministering to human needs. Most of you, I am sure, would give up teaching tomorrow if it weren't for the personal satisfaction of working with youth and seeing the light of understanding dawn for the first time. Yours is a calling to be compared with the ministry and, together, the conscientious teacher, minister and parent mold the pattern of a better tomorrow.

However, the role of the teacher, in some respects, is far different from what it used to be. America is bigger, stronger, more world-minded and more technically alert than it was a few short years ago. This calls for teachers who can take their places in civic affairs, church activities and societies which serve the common good. You may feel like crawling into a hole when the day is over, but this privilege is denied the teacher who takes his job seriously.

But one thing that doesn't change is that the teacher of today must really know his subject, a task becoming more and more difficult. Back home, I have been working with the high-school chemistry teachers of Wilmington. At a recent meeting I apologetically stated that we in industry thought it was more important for a mathematics or chemistry teacher to know his subject than it was to be a learned pedagogue. In other words, the best methods of teaching only brought the best results if the teacher thoroughly knew algebra, trigonometry, geometry or whatever else he was teaching. I noticed a definite reaction within the group, and after the meeting the teachers assured me that industry was mistaken if it believed that the teachers were not deeply concerned with knowing their subjects. In the first place, I was reminded that all teachers must be certified to teach a given subject and beyond this most teachers keep up to date in one way or another. This merely proves again that you can't believe everything you read. The restrained indignation of the teachers who heard my remark was reassuring to all of us because this is a matter of deep concern to America and thus to the world.

Ours is a technological society to a degree that most laymen do not appreciate. Scientifically trained men and women are absolutely essential to almost every aspect of our life. Not only in schools, colleges and research laboratories are the scientists carrying the responsibility of the American way of life, but they are equally important in running our plants, selling the out-

[1] Presented at the annual meeting of The National Council of Teachers of Mathematics, held in Cincinnati, Ohio, April 21–24, 1954.

put of industry, guiding the future of our national defense, prolonging our lives and curing our diseases and contributing immeasurably to the new basic inventions—as well as the gadgets—which everyone has taken for granted. All this calls for well-trained chemists, physicists, biologists, physicians and teachers who can keep America many jumps ahead of our nearest competitors, be they enemy or friend.

May I be so old-fashioned as to say that our present best or the most potent discovery of the future will come from men and women, boys and girls who are carefully trained in the basic disciplines of reading, writing and arithmetic. Actually, there are two basic languages: (a) one's mother tongue, and (b) mathematics. If these are mastered, an American can read, with understanding, anything that is written in English. Without mathematics, almost all the physical sciences are closed to him.

Against this background there appears to be a minority of educators who are anxious to coin a catchy phrase around which they can win personal acclaim. Apparently it is easy to ridicule and difficult to glamorize "reading, writing and arithmetic," but very rewarding to champion such phrases as "The Education of the Whole Child," or "Life Adjustment Education." Such phrases are either meaningless or so obvious that they are meaningless.

In the chemical industry, we find that a college or university graduate who is well founded in the basic principles of chemistry can soon master the details of any assignment. If his program or job is changed, the well-trained chemist can quickly fit into the new job with ease, understanding and competence. Even in this age of specialization, the basic laws of science are the same for the manufacture of gasoline, synthetic fibers, rubber, dyestuffs, insecticides, paint, lubricating oils, soil conditioners, etc. This is just another way of saying that modern chemistry is a science and no longer an art handed from master to apprentice.

But the basic science upon which all sciences are built is mathematics. The physicist and chemist use the same calculus, the civil engineer and navigator use the same trigonometry, the insurance salesman and the housewife use the same arithmetic.

To be a mathematical genius may require a certain gift of God equivalent to those qualities bestowed upon great artists, great musicians, great athletes, great generals and great patriots. But to learn the basic mathematical principles up to and through calculus is within the reach of the average college student who is well trained in each of the preceding disciplines. This is the all-important role of the high-school mathematics teacher. Your job is to so inspire and impress your students that mathematics becomes their ready servant rather than a problem in itself.

Let me illustrate. It is essential that the high-school student learn to be neat, accurate and methodical. There is no better way or place to learn this lesson than through mathematics. He must, for example, be taught that a radical covers everything and nothing more than should be beneath it. He must be taught to place an equal sign opposite the main dividing line of a fraction; he must be neat in making numbers so that there is no difficulty in distinguishing seven from one, three from five, etc. He must be taught accurate mathematical language so that the numerator isn't referred to as the top and the denominator as the bottom. The simple and easy way of writing numbers and their logarithms can be of service throughout one's entire life.

But mathematics is far more than a course in neatness and precision. The reduction of worded problems to numbers and equations is one of the best ways of training the mind that the teaching profession has ever used. I taught trigonometry for several years and solved many trigonometric identities, but have yet to

find a practical use for this knowledge. Nevertheless, this is one of the finest ways of training a student to reason clearly that can be used. On the other hand, the study of trigonometry is relatively simple and easy to teach because it is, in the main, so very practical. It illustrates aptly what a fund of knowledge can be obtained by carefully scrutinizing a simple triangle with its mere three sides and three angles; it is the one branch of mathematics upon which two professions are built.

The inspired teacher can present his subject so that the understanding of new concepts and the opening up of new horizons is a real thrill. The concept of a derivative is one of the greatest contributions to pure reason and clear thinking that the human mind has ever devised in the realm of science. To apply this powerful tool to the solving of maximum-minimum problems is perhaps the apex of a college education. It is almost like handing a student a magic key to the world's storehouse of scientific theory. A workable knowledge of calculus makes the study of almost all college science an easy matter, since this is usually the stumbling block of those students who have trouble with sciences in college.

For example, most chemistry students find that physical chemistry and thermodynamics are the toughest courses they must take. This is simply because both of these studies are based on mathematics—relatively simple calculus at that. Those students who can spend their time studying chemistry, without at the same time trying to comprehend why $\int dx/x = \log x$, have little trouble with modern theoretical chemistry. The toughest derivations are but simple mathematical problems.

Whether the high-school graduate goes on to college, or takes a job, the mastery of high-school mathematics will be of inestimable value throughout his life. This brings me back to where I began. It is the job of the mathematics teacher to insist upon the little but essential things upon which the ultimate knowledge of mathematics is built. You have a marvelous opportunity to do this by laying down a few hard and fast rules to which there are no exceptions. Most of algebra is merely understanding that what you do to one side of an equation you must do to the other. A large segment of trigonometry is contained in the simple and graphic concept of the "unit circle."

You can and should, I think, show that mathematics isn't just cold, impersonal calculation but that there are many practical and personal lessons to be drawn. Show your students, for example, that there is really no irreconcilable difference between plus and minus—it is just a matter of shifting the axis to convert any minus into a plus. So, in life, the proper point of view can change difficulties into opportunities, or failure into success.

You might explain to your students that zero is not nothingness but is a value smaller than anything concerned with that particular problem. Likewise, infinity is not just bigness but is a value larger than anything with which you are working. These concepts should help all of us to see ourselves in the proper perspective, preventing us from being overly egotistical on the one hand and morbidly discouraged on the other.

The simple and exacting operation of balancing an algebraic equation teaches a very practical lesson: what you put of yourself into one pan of life's balance is exactly what you reap on the other side. As you sow, so shall you reap.

Finally, as mathematics teachers, you have the golden opportunity of reversing the unhappy trend that might lead America to mental and moral bankruptcy. Don't make things soft and easy for those who have the capacity and interest to absorb high-school mathematics. There is still no royal road to learning and I am firmly convinced that the inner fortitude and stern stuff that made America strong is still at the heart and core of the boys and girls who sit before you and drink in the lessons you offer from day to day. No

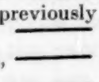
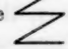
boy gets much of a thrill out of making the varsity team with ease, because that means it is a weak and easily defeated team. Challenge your students, pile on enough homework to make them work hard and remember what they've learned. Know your subject so well that they will instinctively respect and credit you with their subsequent successes. You are on the firing line of a technological America. Keep your sights high so that the students you develop today will be the capable, well-trained, hard working, well-balanced men and women of tomorrow. Be sure that the torch of learning you hand to them will be burning as brightly or brighter than it was when you first took hold of it and dedicated your life to the great and proud profession of teaching and inspiring the youth of America.

From past to present

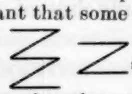
An early system of numeration was recently brought forth in Rex Stout's mystery, "The Zero Clue," which appeared as the second story in his recent trilogy "Three Men Out." And it was necessary for the detective hero, Nero Wolfe, to refer to *Mathematics for the Million* to see what the author, Lancelot Hogben, had to say on the subject before he could solve the case.

The problem was this: a prominent mathematician was found murdered, and this clue, formed by pencils and an eraser was found on his desk. 

What might this mean? It might be a slightly corrupted version of the initials of Nero Wolfe himself. What might be other interpretations of this uncommon representation?

Ancient peoples represented a number by the very same quantity of pebbles or dots; or by sticks or bars. Hogben describes the latter "matchstick" notation of the Chinese in Chapter 7 of the previously mentioned book. Thus, in early China,  would represent three objects or the number three, and would represent two objects or the number two. As the effort was made to speed the formation of these symbols, the writing instrument was not raised from the paper, and they became  and  respectively.

What might the raised dot mean? It is used, of course, as a symbol for multiplication! Therefore, the clue must mean 3 times 2 or 6. Arrest the suspect connected with the number six.

But Chapter 7 mentions a quite different meaning. A need was felt long ago for a convenient system of notation which would represent any number easily regardless of size, and with a minimum of different symbols. (The Roman system is an example of an undesirable one.) This meant that some method was needed to distinguish , that is 32, from other numbers using the same symbols such as 302 or 320. It was the contribution of some unknown Hindu of a dot, ·, which was later 0, to represent the "empty" column.

With this added bit of information, the clue took on new meaning, and the one suspect who was linked with the number 302 was summarily detected and punished.

The use of mathematics as an integral part of a plot is not new in literature. The reader is probably familiar with several. And to be able to follow and understand such applications when they do occur is certainly one of the pleasures that result from a training in mathematics.—*Wendell C. Hall, Myrtle Creek, Oregon*

"Ah, but my Computations, People say,
Reduced the Year to better reckoning?—Nay,
'Twas only striking from the Calendar
Unborn To-morrow, and dead Yesterday."

Omar Khayyám

The use of puzzles in teaching mathematics

JEAN PARKER, *Central High School, Florence, Alabama.*

Puzzles always intrigue high school classes and, hence, form an excellent means of motivating the study of mathematics. Here is an excellent collection of usable puzzles for the high school classroom.

SINCE ANTIQUITY people of all ages have found pleasure in puzzles, tricks, and curiosities of all kinds. The problem of the fox, the goose, and the peck of corn and how to get them across the river was known in the time of Charlemagne, about 800 A.D.[1] The hare and hound problem appears in an Italian arithmetic of 1460, and many other present-day puzzles have come down to us from a much earlier time.[2]

Some of the earlier puzzles have lost much of their original significance or interest inasmuch as the subsequent development of newer branches of mathematics afforded simple solutions which deprived them of their initial mystery and appeal. In other cases, the attention given to these perplexing puzzles ultimately led to an extension or clarification of new mathematical fields. One central fact stands out, however: the appeal of puzzles is as irresistible today as it was two hundred or two thousand years ago.

In the voluminous amount of literature which has been produced concerning the use of puzzles in the classroom, there has been attributed to puzzles every conceivable objective, from thought-provoking recreation to a means of improving attitudes, except as a method of teaching mathematics. Many well-known writers have dealt with this field, including, among others, W. W. R. Ball, H. Dudney, H. Schubert, and S. I. Jones. In general, the consensus seems to be that puzzles are excellent devices for securing the attention of a group, as material for clubs and contests, as part of an enriched program for the bright pupil, as a reward after a certain amount of good work, and for pure enjoyment.

However, there are some few who object to the use of puzzle material in the classroom for any objective. For instance, M. C. Bergen contends that there are only two groups in high school mathematics classes today, the interested and the uninterested.[3] He says, further, that recreations will provide motivation for neither group, since (1) the interested ones need no motivation, and (2) the uninterested ones will certainly get nothing from a problem purposely worded in a tricky and evasive way. If we accept Bergen's hypothesis, we may or may not accept his conclusion. But how can we accept a hypothesis that there are only two classes of mathematics students, when we *know* that there exist varying degrees of interest among pupils. This fact alone tends to invalidate Bergen's statement.

In opposition to Bergen, there are some who believe that a good teacher tries to secure the attention of his students by presenting the subject they are studying in an

[1] Augusta Barnes, "Making Mathematics Interesting," THE MATHEMATICS TEACHER, XVII (November 1921), p. 5.

[2] *Ibid.*

[3] M. C. Bergen, "Misplaced Mathematical Recreations," *School Science and Mathematics*, XXXIX (November 1939), p. 766.

attractive form, and to instill its principles, if possible, by processes agreeable to the student. There are, indeed, some teachers who are willing that their students should be not only interested, but entertained, *if* through such entertainment the *real* work of the classroom can be advanced.

In support of this theory a well-known study by R. B. Porter has been made to determine the effect of the study of mathematical recreations on achievement.[4] Three experiments were carried out, in which the control-group technique was used. The problems used in the experimental group were carefully selected so that they were related to the work studied at the time. The results of these studies indicate that pupils not only may have fun in mathematical recreations of the puzzle nature without any deterring effect upon achievement, but may achieve more in the process. Increase in achievement, however, was only one of many factors in favor of the experimental group as determined by observation of the classes such as:[5]

1. The time spent in recreational activities does not inhibit the covering of a prescribed course content; rather, it provides an opportunity for covering much additional material.

2. The use of recreational material is stimulating to the teacher as well as to the pupil.

3. Recreational items may be synchronized with daily assignments, thereby providing a common meeting ground for student and teacher from which a learning situation may progress.

4. Creativeness is encouraged in the pupil as he devises his own recreational items.

5. A form of research is encouraged through the use of recreational material; pupils who have not previously used library facilities have been observed to do so in search for such items.

[4] R. B. Porter, "Effect of Recreations in Teaching of Mathematics," *School Review*, XLVI (June 1938), p. 423.

[5] *Ibid.*, p. 426.

Certainly any activity which results in the aforementioned advantages would be worth while in itself. But the primary result of this experiment is the conclusive indication that *real mathematics* can be taught through puzzles, wrinkles, and trick problems.

It is my contention that puzzles can be made to serve dual purposes:

1. To secure the interest and attention of the group.

2. To teach mathematics by illustrating and clarifying certain mathematical concepts and techniques, by securing a higher mastery of subject matter, by developing skill in manipulation, by making mathematical learnings more permanent, and by developing an appreciation of the systematic approach of algebraic methods.

It is my purpose to present a sampling of the available puzzle material and methods of using this material in the teaching of mathematics.

An unfailing source of entertainment for the mathematics class is found in the multitude of mathematical problems in which an apparently correct chain of operations leads to an absurd result.[6] These problems also have considerable value in the teaching of mathematics. Not only do they provide a needed relief from day-to-day tasks, but the associations are an aid to memory in recalling important facts which have been temporarily forgotten. No formal statement that division by zero is not permissible will impress the pupil to the same extent that he is shocked when he first sees a "proof" by algebra that $5=7$ or $1=2$.

Almost everyone who has been exposed to elementary algebra has at one time or other seen a proof that any two unequal numbers are equal.

Paradox 1: To prove that any two unequal numbers are equal.[7]

Suppose that

$$a=b+c \qquad (1)$$

[6] C. B. Read, "Mathematical Fallacies," *School Science and Mathematics*, XXXIII (June 1933), p. 585.

[7] E. P. Northrop, *Riddles in Mathematics* (New York: D. Van Nostrand Co., 1944), p. 84.

where a, b, and c are positive numbers. Then inasmuch as a is equal to b plus some other number, a is greater than b. Multiply both sides by $a-b$.

Then

$$a^2-ab=ab+ac-b^2-bc. \qquad (2)$$

Subtract ac from both sides:

$$a^2-ab-ac=ab-b^2-bc. \qquad (3)$$

Factor:

$$a(a-b-c)=b(a-b-c). \qquad (4)$$

Divide both sides by $a-b-c$. Then

$$a=b. \qquad (5)$$

Thus a, which was originally assumed to be greater than b, has been shown to be equal to b.

Paradox 2: To prove any number is equal to any other.[8]

Let a and b be any two given numbers. Let c and d be two equal numbers. Then

$$c=d \qquad (1)$$

$$ac=ad \qquad (2)$$

and

$$bc=bd \qquad (3)$$

$$ac-bc=ad-bd \qquad (4)$$

$$ac-ad=bc-bd \qquad (5)$$

$$a(c-d)=b(c-d) \qquad (6)$$

$$a=b. \qquad (7)$$

These exercises may be introduced when a pupil attempts to divide a number by zero. Now the pupil may ask, "Why can't we divide by zero?" The answer involves the notion of consistency. Division in mathematics is defined by means of multiplication. To divide a by b means to find a number x such that $b \cdot x=a$, whence $x=a/b$. Division by zero then leads either to *no* number or to *any* number. Thus, if we admit zero as a divisor, any two numbers can be shown to be equal. Whether or not we agree that it is important to know division by zero is not allowed, we must agree that here is an opportunity to teach some of the underlying principles of our number system, and to impress students with the "reasonableness" of our definitions, axioms, and postulates (e.g., if division by zero is defined, inconsistencies would occur elsewhere in our laws of operation).

Division by zero is sometimes fairly well disguised. For example in the theory of proportions, it is easy to prove that if two fractions are equal, and if their numerators are equal, then their denominators are equal. That is, from $a/b=a/c$ it can be inferred that $b=c$. This inference is not valid for $a=0$, since step 3 involves division by zero if $a=0$.

Given:

$$a/b=a/c \qquad (1)$$

$$ac=ab \qquad (2)$$

$$c=b. \qquad (3)$$

Consider the following problem in this light. It is desired to solve the equation:[9]

$$(x+5)/(x-7)-5=(4x-40)/(13-x). \qquad (1)$$

Combining terms:

$$[x+5-5(x-7)]/(x-7) = (4x-40)/(13-x). \qquad (2)$$

Simplifying:

$$(4x-40)/(7-x)=(4x-40)/(13-x). \qquad (3)$$

Now since the numerators in (3) are equal, so are the denominators. That is, $7-x=13-x$ or $7=13$.

Now, "How do we know that $4x-40$, the numerator on both sides of (3), is not equal to zero?" Here it should be pointed out that the axioms cannot be applied blindly to equations without taking into consideration the values of the variables for which the equations are true. Thus equation (1) is not an identity which is true for all values of x, but is an equation which is satisfied only for the value $x=10$.

[8] *Ibid.*, p. 86.

[9] *Ibid.*

At this point it would be valuable to introduce problems of this type:

Solve for x:[10]

$$(x-3)(x+3)=x^2-9 \qquad (1)$$

$$(x-2)(x+1)=x^2-x-2. \qquad (2)$$

The fact that x vanishes creates a puzzle which eventually leads up to the difference between an identity and a conditional equation.

Another misuse of the axioms is apparent in the following paradox.[11] In attempting to solve the system of two equations in two unknowns:

$$x+y=1$$

$$x+y=2$$

we are forced to the conclusion that, since 1 and 2 are equal to the same thing, they must be equal to each other, i.e., $1=2$. What is the error? Again, the values of x and y for which both the equations are true must be taken into account and there are no values of x and y for which $x+y=1$, and, at the same time, $x+y=2$. This paradox illustrates a set of inconsistent simultaneous equations in which it is not possible to solve for x and y.

Many other algebraic paradoxes involve the extraction of square root. Using an argument which required division by zero we have already proved that any two unequal numbers are equal to each other. Here is a different proof of the same proposition.

Let a and b be two unequal numbers, and let c be their arithmetic mean, or average.[12]

Then

$$(a+b)/2=c, \quad \text{or} \quad a+b=2c. \qquad (1)$$

Multiply both sides by $a-b$:

$$a^2-b^2=2ac-2bc. \qquad (2)$$

Add $b^2-2ac+c^2$ to both sides:

$$a^2-2ac+c^2=b^2-2bc+c^2 \qquad (3)$$

$$(a-c)^2=(b-c)^2. \qquad (4)$$

Take the square root of both sides. Then

$$a-c=b-c \qquad (5)$$

or

$$a=b. \qquad (6)$$

What is the error? We neglected the fact that a quantity has n nth roots. In passing from step (4) to step (5) only the positive signs are used. Had we written (5) as $a-c=-(b-c)$, we should have obtained our original expression, $a+b=2c$. Here it should be pointed out that the extraction of a square root requires the consideration of both signs, and the one which leads to a contradictory result such as ours must be rejected.

Fallacies are useful in illustrating particular facts. They present those facts in such a manner that they are not easily forgotten. But most important is the fact that fallacies point out the logic and consistency in mathematics. They impress the student with the knowledge that when one obtains absurd results even through seemingly correct operations, there is some logical, concrete explanation which is consistent with the rules of operation.

Geometrical fallacies are very frequently seen. They are usually divided into two types.[13] The first type involves the use of a false theorem; and the second involves the use of an incorrect construction. The flaw in the proof is obvious if the construction is attempted, but if the figure has been prepared in advance, the average geometry class will be greatly puzzled.

The purpose of the following examples is to show the danger of depending too much on the drawing in working out a proof. They illustrate the pitfalls in the faulty construction of figures, i.e., "Relations which appear to the eye to be correct may be erroneous and misleading."[14]

[10] A. R. Meeks, "Recreational Aspects of Mathematics in Junior High School," THE MATHEMATICS TEACHER, XXIX (January 1936), p. 121.

[11] Northrop, *op. cit.*, p. 79.

[12] Northrop, *op. cit.*, p. 81.

[13] Read, *op. cit.*, p. 586.

[14] D. E. Smith and W. D. Reeve, *The Teaching of Junior High School Mathematics* (New York: Ginn and Co., 1927), p. 403.

Paradox 3: Every triangle is isosceles.[15]
Given: any triangle, as ABC.
To prove: ABC is isosceles.

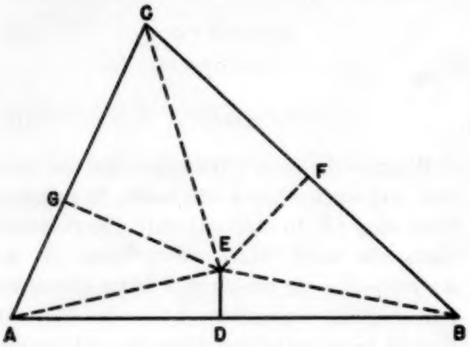


Figure 1

Proof:

Let DE be the perpendicular bisector of AB and CE be the bisector of angle C, meeting DE at E.

From E draw EA and EB.

Draw EG perpendicular to AC and EF perpendicular to CB.

Then

$$\triangle ADE \cong \triangle BDE.$$

Hence

$$AE = BE.$$

$$\triangle CEG \cong \triangle CEF.$$

Hence

$$EG = EF \quad \text{and} \quad CG = CF.$$

Therefore,

$$\triangle AEG \cong \triangle BEF.$$

Hence

$$GA = FB.$$

Since

$$CG = CF$$

it follows that:

$$CG + GA = CF + FB$$

or

$$CA = CB.$$

Therefore ABC is isosceles.

Paradoxes of this type show how easily a logical argument can be swayed by what the eye sees in the figure and so emphasizes the importance of drawing a figure correctly, noting with care the relative position of points essential to the proof.

Paradox 4: To prove that more than one perpendicular may be drawn from a point to a line.[16]

Given: AB and point P outside AB.

Required: To draw two perpendiculars from P to AB.

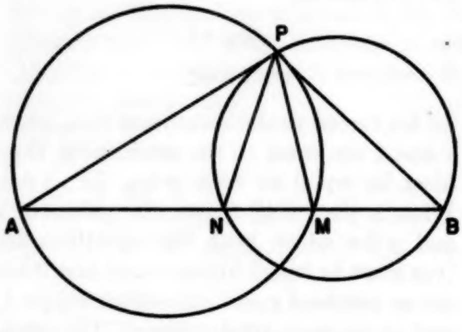


Figure 2

Solution:

Draw AP and PB and construct circles on them as diameters cutting AB at M and N respectively. Then PM and PN will each be perpendicular to AB, since they form right angles inscribed in semicircles.

These examples cannot fail to impress pupils with the importance of careful thinking and *observing* in a logical demonstration. The chief value of such exercises lies in the use that can be made of them to show the need for accurate constructions in geometry. Although important conclusions *may* be deduced from a careful examination of a figure, such cases show that observation unsupported by some reliable check is worthless as a means of proving the truth of geometric statements.

The second type of geometric fallacy illustrates the tendency to misquote a proposition. The absurd results of such

[15] *Ibid.*

[16] Northrop, *op. cit.*, p. 102.

proofs should point out the need for familiarity with theorems, axioms, and postulates that are to be used in a deductive argument.

Paradox 5: Any point on a line is the midpoint of the line.[17]

Given: P any point of AB.

To prove: P is the midpoint of AB.

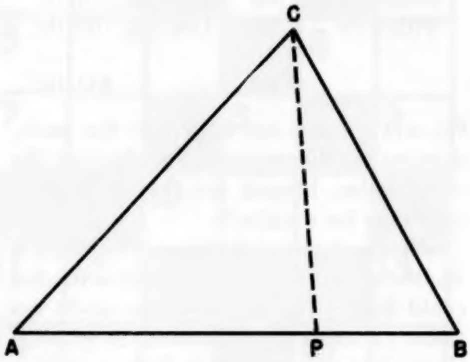


Figure 3

Proof:

On AB as a base construct an isosceles triangle ACB and draw CP. Since $AC=BC$, $CP=CP$, and $\angle A=\angle B$, $\triangle ACP$ and $\triangle BCP$ are congruent, and $AP=PB$ (corresponding parts).

Most boys and girls are thoroughly intrigued by number tricks and age tricks. Much intellectual curiosity and alertness can be aroused in the ordinary boy or girl of high school age through the "mind reading" qualities of the teacher. These problems may be employed in the teaching of mathematics as legitimately and perhaps more effectively as much of the material we are now using, since most of the "mysterious" number problems may be easily explained by the first steps in algebra. They provide needed practice in selecting the essential data from the unessential data, in setting up equations, and in mastering rules of operations and techniques in the solution of equations. Some require very careful reading before they can be translated into the language of simple algebra. This quality will tend to develop what we call "critical" or "logical" thinking. The problems listed below are illustrative of this group.

1. Tell a member of the class to select two numbers and to tell you their quotient and their difference and you will tell him the numbers. How is this done?[18]

The small number is equal to the difference divided by the quotient decreased by one. If $a/b=q$ and $a-b=d$, then it is easily shown by algebra that $b=d/(q-1)$.

2. Tell a person to think of a number, to square it, to square the next larger number, and then tell you the difference between the squares and you will tell him the number. How is it done?[19]

Let n be the number and $n+1$ the next larger; then the squares are n^2 and n^2+2n+1. The difference is $2n+1$. Hence, subtract 1 and divide by two and you have the number selected.

Puzzles of this type emphasize the advantages of the systematic approach to the solution of problems that is found in algebra.

The ability to square numbers mentally is another elementary algebraic puzzle. Its algebraic explanation affords a highly desirable application of the formulas to arithmetical computation, while at the same time the algebraical statement extends the rule to all numbers between 25 and 150.[20] Rules of this kind are excellent for rapid calculation.

The rules are all applications of the simple formulas $(a+b)^2=a^2+2ab+b^2$ and $(a-b)^2=a^2-2ab+b^2$ and the teacher of algebra should emphasize this kind of simple application of algebraic formulas. Thus

$$(50-a)^2=2500-100a+a^2$$

$$(50+a)=2500+100a+a^2$$

$$(100+a)=10{,}000+200a+a^2$$

[17] Smith and Reeve, *op. cit.*, p. 402.

[18] *Ibid.*, p. 395.

[19] *Ibid.*, p. 394.

[20] L. C. Karpinski, "A Rule to Square Numbers Mentally," *School Science and Mathematics*, XV (January 1915), p. 20.

$$= 100(100+a+a)+a^2$$

$$(20+a) = 400+40a+a^2$$
$$= 20(20+a+a)+a^2$$

$$(10+a) = 100+20a+a^2$$
$$= 10(10+a+a)+a^2.$$

Another type of puzzle is that frequently found in algebra books under the title of "practical problems." In most puzzles of this kind, the difficulty lies in the phraseology, while the mathematics is simple. They provide drill in setting down facts in a logical sequence and also teach the solution of equations through problems made interesting by their puzzling nature.

For example:

1. A man is twice as old as his wife was when he was as old as she is now. When she is as old as he is now, the sum of their ages will be 100 years. Find their ages now.[21]

The problem involves two equations:

$$m = 2(w-m-w)$$

and

$$m+m-w+w+m-w = 100$$
$$m = 44\tfrac{4}{9} \qquad w = 33\tfrac{1}{3}.$$

2. Mary is 18 years old. Ann is twice as old as Mary was when Ann was as old as Mary is now. How old is Ann?[22]

Let us say that x years ago Ann was as old as Mary is now; that is, she was then 18.

Then

$$a-x = 18$$

and also

$$a = 2(18-x)$$

hence

$$a = 18+x$$

and so

$$18+x = 2(18-x)$$

solving,

$$x = 6$$

hence

$$a = 2(18-6) = 24.$$

[21] Smith and Reeve, *op. cit.*, p. 399.
[22] *Ibid.*, p. 399.

Perhaps the puzzle most frequently discussed by the public is the "trick" puzzle. It is said that the following puzzle caused chaos in banking circles:

Assume we make a deposit of $50 in a bank.

Withdraw	$20.00	Leaving	$30.00
Withdraw	15.00	Leaving	15.00
Withdraw	9.00	Leaving	6.00
Withdraw	6.00	Leaving	00.00
	$50.00		$51.00

We now present our figures to the bank, showing the discrepancy, and demand the extra dollar. Repeat ten thousand times and retire for a while.[23]

Since such problems are so common it would be well if the mathematics teacher could find a way in which she might use them to teach mathematics. In general, these discrepancies result from a lack of observation and critical thinking. A puzzle of this type should arouse the innate curiosity of most high school pupils and should develop (1) the processes of thinking and (2) a questioning attitude toward such misleading results. (Why should the second column add up to the same amount as the first?)

There are many mechanical puzzles, such as the "Fifteen Puzzle" and the "Tower of Hanoi" which have a mathematical basis far above the level of thinking in the secondary school. However, there is enough enjoyment in the interesting stories accompanying these puzzles and in the manipulation of them knowing that there is a mathematical explanation, to warrant their use in the classroom for recreational purposes. It is to be hoped that through such recreational devices as these a true interest in mathematics may be furthered and a desire to continue mathematical study may be developed through the realization that *mathematics can be fun.*

[23] Harriet Heald, *Mathematical Puzzles* (Washington Service Bureau, 1217 13th Street N.W., Washington, D.C.), p. 6.

The popularity of the daily newspaper crossword puzzle has probably caused the development of the cross-number puzzle as illustrated below.

AREAS AND PERIMETERS[24]

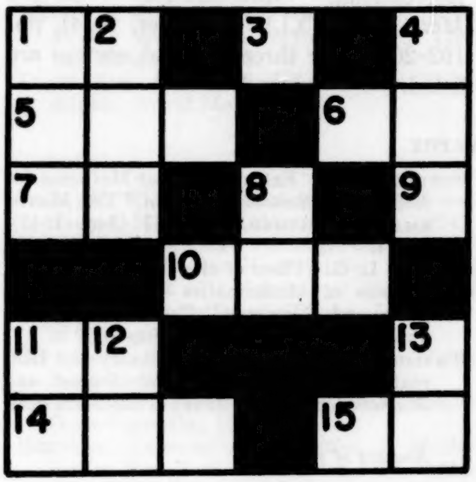


Figure 4

Horizontal

1. Perimeter of a 9″ square.
3. Area of a 1″ square.
5. Area of a rug 9′×12′.
6. Circumference of a circle with a diameter of 7″.
7. Area of a rectangle 16″×3$\frac{1}{2}$″.
8. How many sides has a square?
10. Find the area of a triangle with a base of 16″ and an altitude of 14″.
11. Same definition as (11) vertical.
14. Perimeter of a wheel 49″ in diameter.
15. How many 2″ squares can be made to fit into a rectangle 4″ by 10″?

Vertical

1. Area of a room 30′×10$\frac{1}{2}$′.
2. Area of a field 60 rd.×10.1 rd.
3. Area of a unit square.
4. Area of a square 15 cm. on a side.
8. Perimeter of a fence 12 yd.×8$\frac{1}{2}$ yd.
11. Perimeter of a triangle which is 7″ on each side.
12. Perimeter of a rectangle 4.8 cm. ×2.7 cm.
13. Perimeter of a lawn which measures 23$\frac{3}{4}$ ft. in length and 21$\frac{1}{4}$ ft. in width.

The purpose of the cross-number puzzle is its use as a motivating device which will enable the teacher to re-create interest in solving mathematical problems. It involves the construction of a puzzle which the students can complete by writing in the correct answers to a set of problem exercises. Its mathematical value lies in the interesting manner in which a review of a unit of work may be undertaken and in its use in the regular classroom instead of traditional tests and written exercises.

It appears that we may safely conclude that most of us have a "puzzle instinct" and that the puzzle question at the right time and place will not only make a class more interesting, but can further the primary responsibility of the teacher, i.e., teaching mathematics. Specific instances have been pointed out in which it is possible both to motivate the work and at the same time insure its mastery. Puzzles and paradoxes lend themselves to the teaching of (1) specific facts, e.g., division by zero is not permitted; (2) techniques in problem solving, e.g., selecting pertinent data, establishing equalities, and operations with equalities; (3) basic mathematical concepts, e.g., accuracy in drawings, and familiarity with well-known propositions in geometry; (4) critical thinking; (5) an appreciation of the power of mathematics.

Puzzle material, properly related to class work can, therefore, be made a valuable aid to teaching.

Now, should the teacher of mathematics desire to locate additional material concerning puzzles, the following sources are suggested:

1. Several newspapers and magazines have daily and weekly features of mathe-

[24] E. J. Berger, "Devices for a Mathematical Laboratory," THE MATHEMATICS TEACHER, XLII (January 1951), p. 33.

matical puzzles (e.g., *Popular Science* puzzle page).

2. Mathematical publications as THE MATHEMATICS TEACHER, *School Science and Mathematics, Scripta Mathematica,* and *American Mathematical Monthly.*

3. The bibliography listed in this article.

4. The pupils themselves. With guidance and direction, pupils will submit many interesting items.

5. The teacher. With an underlying source of error of a mathematical principle as a basis, endless problems can be created by the teacher who is interested in this type of work.

6. Schaaf, N. L., "List of Works on Recreational Mathematics," *Scripta Mathematica,* XLIV (October, 1944), pp. 192–200. Over three hundred sources are listed.

BIBLIOGRAPHY

Articles on the use of puzzles in the classroom

BAKST, A. "Recreational Mathematics," THE MATHEMATICS TEACHER, XLIII (Oct. 1950), 347, 416–417.

BARKAN, S. H. "Puzzle Instinct in Teaching Mathematics," *B. H. Points,* XII (Oct. 1930), 69–70.

BARNES, AUGUSTA. "Making Mathematics Interesting," THE MATHEMATICS TEACHER, XVII (Nov. 1921), 4–10.

BERGEN, M. C. "Misplaced Mathematical Recreations," *School Science and Mathematics,* XXXII (Nov. 1939), 766–768.

BRANDES, L. G. "Mathematics Can Be Fun," *Clearing House,* XXV (Oct. 1950), 75–79.

CAJORI, F. "Absurdities Due to Division by Zero," THE MATHEMATICS TEACHER, XXII (Oct. 1929), 366–368.

HASSLER, J. O. "What Price Enrichment?" THE MATHEMATICS TEACHER, XXXIV (Nov. 1941), 243–247.

KARPINSKI, L. C. "A Rule to Square Numbers Mentally," *School Science and Mathematics,* XV (Jan. 1915), 20–21.

LEACOCK, S. B. "Human Interest Put into Mathematics," THE MATHEMATICS TEACHER, XXII (May 1929), 302–304.

MEEKS, A. R. "Recreational Aspects of Mathematics in Junior High School," THE MATHEMATICS TEACHER, XXIX (Jan. 1936), 20–22.

MOORE, LILLIAN. "The Challenge of the Bright Pupil," THE MATHEMATICS TEACHER, XXXIV (June 1941), 155–157.

NEWHALL, C. W. "Recreations in Secondary Mathematics," *School Science and Mathematics,* XV (April 1915), 277–293.

PORTER, R. B. "Effect of Recreations in Teaching of Mathematics," *School Review,* XLVI (June 1938), 423–427.

READ, C. B. "Mathematics Magic," *School Science and Mathematics,* XXXVII (May 1937), 597.

———. "Mathematical Fallacies," *School Science and Mathematics,* XXXIII (June 1933), 585–589, also 977–983.

SAUERBRAU, M. A. "Mathematical Recreation," *Clearing House,* V (Feb. 1931), 373–374.

SCHULTE, M. L. "Extra-Curricular Mathematics Activities in Secondary School," THE MATHEMATICS TEACHER, XXXIII (Jan. 1940), 32–34.

SIMONS, L. G. "Place of the History and Recreations of Mathematics in Teaching Algebra and Geometry," THE MATHEMATICS TEACHER, XVI (Jan. 1923), 94–101.

TAYLOR, S. H. "Mathematics Library and Recreational Programs," *School Science and Mathematics,* XXX (June 1930), 626–634.

Sources of puzzles

PERIODICALS

BAKST, A. "Recreational Mathematics," THE MATHEMATICS TEACHER, XLIII (Oct. 1950), 347, 416–417.

BENTLEY, B. R. "Recreations for Mathematics Club," THE MATHEMATICS TEACHER, XXIII (Feb. 1930), 95–103.

BERGER, E. J. "Devices for a Mathematics Laboratory," THE MATHEMATICS TEACHER, XLIV (Jan. 1951), 31–34.

———. "Devices for a Mathematics Laboratory," THE MATHEMATICS TEACHER, XLIII (Oct. 1950), 287–289.

CLARKE, E. H. "Prize Problems for Prize Students," THE MATHEMATICS TEACHER, XXIII (Jan. 1930), 30–34.

CAJORI, F. "Absurdities Due to Division by Zero," THE MATHEMATICS TEACHER, XXII (Oct. 1929), 366–368.

JONES, H. I. "What Day Is It?" *School Science and Mathematics,* XXIII (May 1923), 825–830.

PIERCE, MARTHA. "Mathematical Recreations," THE MATHEMATICS TEACHER, XIX (Jan. 1926), 13–24.

RADIUS, C. "Problems for Recreations," *School Science and Mathematics,* XXXIV (Jan. 1934), 87–90.

READ, C. B. "Mathematical Fallacies," *School Science and Mathematics,* XXXIII (June 1933), 585–589, also 977–983.

SMITH, J. R. "Cross-figure Puzzles," *School Science and Mathematics,* XLV (June 1945), 576–578.

PAMPHLETS

SMITH, D. E. *Numbers and Numerals.* New York: Teachers College, Columbia University. 50¢.

———. Kit M, *Brain Resters and Testers.* Delaware, O.: Cooperative Recreation Service. 25¢.

———. Kit U, *Puzzle Craft.* Delaware, O.: Cooperative Recreation Service. 25¢.

HEALD, H. *Mathematical Puzzles.* Washington, D. C.: Washington Service Bureau, 1217 13th Street N.W. 15¢.

LLOYD, SAM. *A Puzzle Book for Children.* Philadelphia: David McKay Co.

BOOKS

BALL, W. W. R. *Mathematical Recreations and Essays.* New York: Macmillan Co., 1947.

BAKST, A. *Mathematics, Its Magic and Mastery.* New York: D. Van Nostrand Co., 1941.

BRESLICH, E. R. *The Teaching of Mathematics in the Secondary School.* Chicago: University of Chicago Press, 1930.

COLLINS, A. F. *Fun With Figures.* New York: Appleton-Century Co., 1938.

DE MORGAN, A. *Budget of Paradoxes.* Chicago: Open Court Co., 1915.

DRESDEN, ARNOLD. *An Invitation to Mathematics.* New York: H. Holt and Co., 1936.

DUDENEY, H. E. *Amusements in Mathematics.* New York: Thomas Nelson and Sons, 1917.

———. *Canterbury Puzzles.* New York: Thomas Nelson and Sons, 1927.

———. *Puzzles and Curious Problems.* New York: Thomas Nelson and Sons, 1932.

HEATH, R. V. *Mathemagic.* New York: Simon and Schuster, 1933.

JONES, S. I. *Mathematical Clubs and Recreations.* Nashville: S. I. Jones Co., 1940.

———. *Mathematical Nuts.* Nashville: S. I. Jones Co., 1923.

———. *Mathematical Wrinkles.* Nashville: S. I. Jones Co., 1923.

KASNER, E. and NEWMAN, J. *Mathematics and the Imagination.* New York: Simon and Schuster, 1940.

KRAITCHIK, M. *Mathematical Recreations.* New York: W. W. Norton and Co., 1942.

LICKS, H. E. *Recreation in Mathematics.* New York: D. Van Nostrand Co., 1917.

MERRILL, H. A. *Mathematical Excursions.* Norwood, Mass.: Norwood Press, 1933.

NORTHROP, E. P. *Riddles in Mathematics.* New York: D. Van Nostrand Co., 1944.

SMITH, D. E. *The Wonderful Wonders of One-Two-Three.* New York: McFarlane, Ward, McFarlane, 1937.

———. *Number Stories of Long Ago.* Boston: Ginn and Co., 1919.

——— and REEVE, W. D. *The Teaching of Junior High School Mathematics.* New York: Ginn and Co., 1927.

SCHUBERT, H. *Mathematical Essays and Recreations.* Chicago: Open Court Co., 1918.

VENTRESS, H. E. *Mathematical Puzzles and Problems.* Washington, D. C.: Washington Bureau.

WEEKS, RAYMOND. *The Boy's Own Arithmetic.* New York: E. P. Dutton and Co., 1924.

WHITE, W. F. *A Scrapbook of Elementary Mathematics*, Chicago: Open Court Co., 1910.

Are postulates in a mathematical system "true"?

If you want to use a mathematical system in a very practical way, such as in surveying and astronomy, must its principles agree with the reality that you observe and measure? The following excerpts will appear startling to those who answer "yes" to these questions:

"To simplify the computations necessary for the determinations of the direction of the meridian, of latitude, and of longitude or time, certain concepts of the heavens have been generally adopted. They are the following:

a) The earth is stationary.
b) The heavenly bodies have been projected outward, along lines which extend from the center of the earth, to a sphere of infinite radius called the celestial sphere.

The celestial sphere has the following characteristics:

a) Its center is at the center of the earth.
b) Its equator is on the projection of the earth's equator.
c) With respect to the earth, the celestial sphere rotates from east to west about a line which coincides with the earth's axis. Accordingly, the poles of the celestial sphere are at the prolongations of the earth's poles.
d) The speed of rotation of the celestial sphere is 360° 59.15′ per 24 hours."[1]

[1] Keuffel and Esser Company, *Solar Ephemeris and Surveying Instrument Manual*, 1954, p. 56.

Common errors made in general mathematics by high school students of Louisiana

ROGERS E. RANDALL, *Southern University, Baton Rouge, Louisiana.*

Evidence continues to accumulate that our high school graduates do not know how to compute; however, in striving to improve computation, care should be taken that this does not overshadow the more important mathematical goals in our high school mathematics classes.

BECAUSE OF the generally poor performance of students in the mathematical computations required in a physical science course taught by the writer, he set out to learn the extent of the mathematical literacy of Louisiana's Negro high school students in the areas of arithmetic and algebra. Specifically, he wished to determine (1) the common errors made by high school students enrolled presently in a general mathematics course, and (2) to what extent these students had the ability to solve word problems.

PROCEDURE

As a part of the competitive testing of the Louisiana Interscholastic Athletic Literary Association for the second semester of the school year 1954, an outline covering topics to be taught was mailed to each participating school during the first semester of the academic year 1953–54.

Dr. Leroy R. Posey, Jr., Head of the Mathematics and Physics Department of Southern University, constructed a mastery test in general mathematics, which was administered to the best students enrolled in a general mathematics course in their respective high schools. The students were selected by their respective mathematics teachers. The time period for the test was two hours.

The mastery test referred to above included problems involving the mathematical operations of addition (decimals, fractions, mixed numbers, and conversions), subtraction (decimals, mixed numbers, fractions, and conversions), division (decimals, whole numbers, and mixed numbers), multiplication (decimals and mixed numbers), solving linear equations, factoring, and solving word problems.

One hundred thirty-one students from fifty-six high schools, composed of 65 female students and 66 male students, participated in the general mathematics competitive testing program in the state.

RESULTS

The results of this test in terms of scores reveal a score range for females of 0–70; for males, 0–96.6.[1] The mean for the female students is 33.3; for the male students, 37.6.[2]

[1] Scores based on 100 per cent.
[2] No data were secured that would give a reason for a better performance from the male students.

TABLE 1.—Types of Problems and Common Errors

Problem number and process	Problems	Problem number	Common errors made
I. Addition	(a) 0.00214, 4.32, 4.6009, and 174.01018 (b) 23 2/3, 5 1/8, 6 3/4, 5/12 (c) 1/6, 5/9, 7/30, and 2/3 (d) 4 yards, 2 feet, and 7 inches (Express the result in feet) (e) 5.42 kilometers, 156.3 meters, 56.8 centimeters, and 8 millimeters (Express results in meters)	I & II	1. Wrong position of decimal point 2. Changing numbers to fractions 3. Adding and subtracting fractions 4. Changing from one unit to another 5. Subtracting larger number from smaller number
II. Subtraction (subtract the first number from the second)	(a) 5.007, 12.02 (b) 2 7/8, 5 1/4 (c) 9/25, 8/15 (d) 5.2 centimeters, 1.3 meters (Express result in centimeters) (e) 3 1/2 feet, 5 1/4 yards (Express result in yards)		
III. Division	(a) 0.510224 by 0.104 (b) 5.83 by 58300 (c) 202,570 by 235 (d) 2 5/8 by 5 1/3 (e) 0.003 by 15	III III & IV	1. Dividing and not placing zeros in answer 1. Changing mixed number to correct fraction 2. Placing decimal in wrong place
IV. Multiplication	(a) 2.05×0.6 (b) $77.7 \times .0009$ (c) $.0606 \times .2002$ (d) $9\ 5/7 \times 3\ 3/5$		
V. Linear equations	(a) $5c+9c-6c=132$ (b) $7(g-8)-6(2g-15)=4(6-g)$ (c) $3s/2-7/3=4s/5-7/5$ (d) $3d-5d/6=d/2+5/9$	V & VI	1. Simplifying linear equations 2. Removing denominator 3. Factoring 4. Solving for unknown
VI. Factoring	(a) $4m^2r^2+8m^2r-60m^2$ (b) $9x^2-16y^2$ (c) $15x^2+14x-8$ (d) $ax^2-2ax-35a$		
VII. Word problems	(a) The sugar contained in the sugar beet is 6 1/4% of the weight of the beet. How many pounds of beets will be required to produce 175 pounds of sugar? (b) A lady buys a piece of cloth containing 48 yards at 22¢ a yard. She uses 33 yards for various purposes, and the rest for making four skirts. What did the cloth cost per skirt? (c) In a certain triangle having three unequal angles, the smallest angle is 1/6 the largest, and the remaining angle is 1/3 the largest. Compute the size of each of the angles of the triangle.	VII	1. Setting word problems up correctly 2. Changing per cent to decimal

Table 1 shows the types of problems and common errors made on the test.

Table 2 shows the number and per cent of correct and incorrect responses and problems not attempted. The principal difficulty in decimal problems proved to be placement of the decimal point. A large number of the students did not attempt to solve word problems. This may suggest their inability to read and interpret word problems. The principal difficulty in solving problems involving factoring appeared to be that the majority of the students just did not know anything about the processes involved in factoring. The difficulty in solving problems containing linear equations seemed to be that the majority of the students did not know algebra.

Conclusions and recommendations

An examination of the data in Table 2 revealed that:

1. There were signs of carelessness in multiplication, addition, subtraction, and division. (Common errors shown in Table 1.)

TABLE 2.—Number and Per Cent of Correct Responses, Incorrect Responses and Problems Not Attempted

Problem numbers[3]	Correct responses		Incorrect responses		No attempts	
	Number	*Per Cent*	*Number*	*Per Cent*	*Number*	*Per Cent*
I (a)	83	63.3	47	35.8	1	.77
(b)	70	53.6	57	43.5	4	3.05
(c)	46	35.1	76	58.01	9	6.8
(d)	65	49.6	50	38.1	16	12.2
(e)	19	14.5	48	36.6	64	48.8
II (a)	90	68.7	36	27.4	5	3.8
(b)	87	66.4	36	27.4	8	6.1
(c)	78	59.5	40	30.5	13	9.9
(d)	23	17.5	61	46.5	47	35.8
(e)	27	20.6	77	58.7	27	20.6
III (a)	70	53.6	50	38.1	11	8.3
(b)	57	53.5	49	37.4	25	19.08
(c)	82	62.5	34	25.9	15	11.4
(d)	46	35.1	63	48.09	22	16.7
(e)	56	42.7	39	29.7	36	27.4
IV (a)	81	61.8	37	28.2	13	9.9
(b)	72	54.9	39	29.7	20	15.2
(c)	73	55.9	41	31.2	17	12.9
(d)	51	38.9	53	40.4	27	20.6
V (a)	60	45.8	50	38.1	21	16.03
(b)	26	19.8	69	52.6	36	27.4
(c)	10	7.63	76	58.01	45	34.3
(d)	11	8.3	61	46.5	59	45.03
VI (a)	3	2.2	68	51.9	60	45.8
(b)	19	14.5	50	38.1	62	47.3
(c)	7	5.3	56	42.7	68	51.9
(d)	14	10.6	51	38.9	66	50.3
VII (a)	69	52.6	44	33.5	18	13.7
(b)	4	3.05	82	62.5	45	34.3
(c)	9	6.8	41	31.2	81	61.8

[3] See Table 1 for number and description of problems.

2. A large number of students did not know how to handle the decimal point correctly in addition, multiplication, subtraction and division.

3. A large percentage of the students did not know how to handle denominate numbers correctly. See problems I(d), I(e), II(d), and II(e).

4. A large number of students did not know how to change mixed numbers to fractions. See problems III(d) and IV(d).

In light of the results and conclusions of this study, it seems justifiable to recommend that:

1. Competence in arithmetic be stressed among high school students.

2. A great deal of mathematical thinking be stressed on the part of the student and teacher (word problems may serve well at this point).

3. Emphasis be placed on accuracy and speed in learning the skill operations, but that accuracy be stressed most.

4. Teachers of mathematics in the State of Louisiana make concerted and conscious efforts to improve the teaching and learning of general mathematics by raising the existing academic standards in teaching the subject to high school students.

What's new?

BOOKS

SECONDARY

Algebra, Course 1, Howard F. Fehr, Walter H. Carnahan, Max Beberman, Boston, Massachusetts, D. C. Heath and Company, 1955. Cloth, xi+484 pp., $3.00.

Algebra, Course 2, Howard F. Fehr, Walter H. Carnahan, Max Beberman, Boston, Massachusetts, D. C. Heath and Company, 1955. Cloth, x+502 pp., $3.00.

Functional Mathematics, Book 3, William A. Gager, Lilla C. Lyle, Carl N. Shuster, Franklin W. Kokomoor, New York, Charles Scribner's Sons, 1955. Cloth, xiii+481 pp., $3.20.

Mathematics, A Third Course, Myron F. Rosskopf, Harold D. Aten, William Reeve, New York, McGraw-Hill Book Company, Inc., 1955. Cloth, iii+438 pp., $3.48.

Plane Geometry, Arthur F. Leary and Carl N. Shuster, New York, Charles Scribner's Sons, 1955. Cloth, ix+510 pp., $3.80.

COLLEGE

Statistical Methods for the Behavioral Sciences, Allen L. Edwards, New York, Rinehart and Company, Inc., 1954. Cloth, xvii+542 pp., $6.50.

MISCELLANEOUS

Numbers: Fun and Facts, J. Newton Friend, New York, Charles Scribner's Sons, 1954. Cloth, xi+208 pp., $2.75.

Public Education and the Future of America, The Educational Policies Commission, Washington, D. C., National Educational Association, 1955. Paper, vi+98 pp., $1.50.

What Every Woman Should Know About Finance, Mabel Raef Putnam, New York, Charles Scribner's Sons, 1954. Cloth, xiii+254 pp., $3.50.

BOOKLET

Money Management, Your Budget (Revised, 1954), Consumer Education Department, Household Finance Corporation, 919 North Michigan Avenue, Chicago 11, Illinois. Illustrated booklet, 36 pp., 10¢.

EQUIPMENT

Slated Globes, Denoyer-Geppert Company, 5235 Ravenswood Avenue, Chicago 40, Illinois. Slated globes in wood cradles; available in 6″, 8″, 12″, 16″, 20″, and 24″ sizes; 8″ globe, $4.25 plus postage; 12″ globe, $22.00 plus express charges.

Some arithmetical fundamentals of value to junior high school teachers

J. MAURICE KINGSTON, *University of Washington, Seattle, Washington.*

A discussion of some elementary points in mathematics so frequently misunderstood by teachers and of great import for the student's future success in the study of algebra

WE HAVE ALL been confronted by numerous cases of people who either say they hate arithmetic, or are sure they will not be able to do algebra, or say they never could understand algebra. I feel that to a greater or lesser extent these problem situations are related and that if we can do something effective to reduce the number in the first group, greater success will be achieved with the latter two groups.

To be specific, I believe, for one thing, it would be of great value to reduce to an absolute minimum the amount of memorized procedures which are still sometimes imposed on children. By so doing I believe the actual arithmetic taught need not be reduced nor do I imply that we are to eliminate a reasonable amount of practice. Let me say at once that the current trend toward emphasizing understanding and meaning in arithmetic is a major step in this direction. Another step which I feel would be beneficial would be to overcome our fear of using simple algebraic techniques, where they can be of particular assistance in arithmetic without requiring elaborate preparation. I believe the traditional algebra can be made a much more natural generalization of arithmetic than is now usually the case and the transition from one to the other should then be accomplished without any shock. It is my opinion that if we fail to offer children this opportunity, we are in danger of cheating them. To say that it will serve only to confuse them is to underestimate their capabilities in many cases and to yield to the temptation to resist any alterations in procedures which "seem to work pretty well." Hence, in what follows, the reader will not be surprised at the algebraic flavor which will be evident and which, it seems to me, can add much to the elegance of the subject.

I believe that there is a relatively small number of basic principles associated with our number system of arithmetic, which, when fuller use is made of them, will lead to greater understanding and improved performance. In the first portion of this discussion the references will be of a practical and basic type while the latter part will concern itself with more abstract notions.

Let us note at the beginning that equality is a proper equivalence relation and hence possesses the reflexive, transitive and symmetric properties. I should like to direct our attention particularly to the symmetric property which guarantees that $a=b$ implies $b=a$. Although this fact is not unfamiliar I believe we might well emphasize it more strongly as early as, for example, when multiplication is first encountered. The fact that $15=3\times 5$ is going to play an equally important role beside its symmetric twin, $3\times 5=15$, for example, when we wish to add $\frac{1}{12}$ and $\frac{1}{15}$. If children have become accustomed to the desirable habit of "splitting" numbers

into their factors, the "common denominator process" will be simpler to understand and use. The argument will proceed logically that we need the individual factors 3, 4 and 5. Let us suppose that our addition problem is somewhat more involved, say, $\frac{1}{12}+\frac{1}{18}+\frac{1}{27}$. If we note that $12=3\times2\times2$ while $18=3\times3\times2$ and $27=3\times3\times3$, then the smallest number which will contain 12, 18 and 27 must contain the necessary factors, i.e., $3\times2\times2\times3\times3=108$. However, when it comes to changing $\frac{1}{18}$ into the correct multiple of $\frac{1}{108}$, we do more work than necessary if we neglect the factored forms of 108 and 18 which we have just obtained. The easiest way to divide 108 by 18 is to remove the factors of 18 from the factors of 108, leaving 2×3 or 6. Similarly the factors 2×2 remain when 108 is divided by 27. The addition follows easily:

$$\frac{1}{12}+\frac{1}{18}+\frac{1}{27}=\frac{9}{108}+\frac{6}{108}+\frac{4}{108}=\frac{19}{108}\cdot$$

In this connection the commutativity of multiplication is much more than a minor property as we shall see later.

The integer *one* is not usually given, it seems to me, an adequate opportunity in elementary arithmetic to demonstrate its capabilities as a multiplier. Consider the problem of "cancellation" in the reduction of fractions. In some treatments the reduction is based on the assumed correctness of dividing the numerator and denominator by the same non-zero number. To me this has always seemed a weak approach and one requiring considerably more justification than that usually given. I should think a child's reaction might well be: Why divide rather than subtract equal numbers in order to reduce $\frac{12}{15}$? On the other hand let us see if the number *one* cannot make the situation clear. Let us assume that the multiplication of fractions has been defined and the fact observed that $\frac{2}{2}=\frac{3}{3}=\frac{4}{4}=\cdots=1$. Then if we are accustomed to factoring, the fraction $\frac{12}{15}$ is seen immediately as $\frac{3\times4}{5\times3}$ or, by invoking commutativity, as $\frac{4\times3}{5\times3}\cdot$ The symmetric characteristic of equality again makes an appearance, this time in the definition of the product of two fractions, allowing us to work from either side of the equation $\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}\cdot$ This gives us $\frac{4}{5}\times\frac{3}{3}=\frac{4}{5}\times1$ which must be $\frac{4}{5}$ on account of that delightful property of the number *one*. At this point a teacher, if he wishes, may accelerate the procedure by introducing a cancellation operation. (Lest I might be misunderstood may I say that I realize and fully approve that reduction of fractions will have been previously treated graphically and with the use of concrete materials.) I believe this treatment in arithmetic will be a strong deterrent when a student is tempted to put the answer *zero* when "everything has cancelled out." Too, he will not be so likely to want to cancel c in the fraction $\frac{ab+c}{c^2}$ when he studies algebra. We have seen that multiplication by the number *one* occurs both in "reducing" and in "expanding" a fraction, if such poor words can be countenanced in connection with operations which leave the value of a fraction unchanged. It is the same step which guarantees in long division of decimals that we can always work with a whole number as a divisor.

Another situation in which the number *one* plays a similarly effective part is in justifying or replacing the "invert and multiply" rule. Let us consider the so-called complex fraction $\frac{\frac{2}{3}}{\frac{3}{4}}\cdot$ We know that any non-zero number divided by itself is equal to *one* and hence we are free to

choose the representation of *one* which is most useful for our purpose. In this case the choice is directed by our desire to create *one* in the denominator of the original fraction. This can be accomplished by multiplication in the denominator by $\frac{4}{3}$. Therefore to multiply by *one* we will use the form $\frac{\frac{4}{3}}{\frac{4}{3}}\cdot$ The result is then

$$\frac{\frac{2}{3}}{\frac{3}{4}}\times\frac{\frac{4}{3}}{\frac{4}{3}}=\frac{\frac{2}{3}\times\frac{4}{3}}{\frac{3}{4}\times\frac{4}{3}}=\frac{\frac{2}{3}\times\frac{4}{3}}{1}=\frac{2}{3}\times\frac{4}{3}$$

and our "invert and multiply" rule is completely evident. There is another procedure of a similar type which is also very effective, as follows:

$$\frac{\frac{2}{3}}{\frac{3}{4}}=\frac{\frac{2}{3}}{\frac{3}{4}}\times\frac{12}{12}=\frac{8}{9}\cdot$$

Whether or not such problems are now considered "proper" to give to children, I should at least like to be certain that teachers know how to handle them easily and with complete understanding. In this vein, let us deal with one that is more complex, e.g. $\frac{\frac{1}{3}-\frac{1}{7}}{\frac{1}{4}+\frac{1}{5}}\cdot$ There is, of course, the possibility of carrying out two "common denominator procedures" followed by an application of the "invert and multiply" rule. But suppose that we simply multiply our fraction by $\frac{3\times4\times5\times7}{3\times4\times5\times7}\cdot$ This will give

$$\frac{(4\times5\times7)-(3\times4\times5)}{(3\times5\times7)+(3\times4\times7)}=\frac{140-60}{105+84}=\frac{80}{189}\cdot$$

Perhaps if we demonstrate some elegance in arithmetic we may avoid making it appear like a chore.

I believe it would be profitable to incorporate into arithmetic quite early the use of parentheses. I have not forgotten the elementary teacher-trainee who was confronted with the following objective question on one of my tests. I set up a long division symbolically, thus:

$$\begin{array}{r} S \\ W)\overline{F} \\ G \\ \hline T \end{array}$$

and asked that the correct statement be chosen from five given statements of the form: W multiplied by S and the result added to T gives F. His verbal reaction to this question was, "I can't do it; this is algebra." I would like to see the parenthesized form $(a\times b)+c-(d\div f)$ replace the necessity for remembering the rule that multiplications and divisions must precede additions and subtractions in such expressions as $a\times b+c-d\div f$. Then $F=W(S)+T$ would be associated with the operation of division directly from its introduction and the alternate form $\frac{F}{W}=S+\frac{T}{W}$ could be obtained easily.

The question many teachers dread is "Why can't we divide by *zero*?" This question need cause no difficulty if we realize that the ability to carry out a division procedure depends on the existence of a quotient. Returning to $F=W(S)+T$, it is easy to observe that since T is smaller than F, there is no quotient S which will act when W is *zero*. This should not be a source of confusion since it is only one of the special characteristics of *zero*, a number which plays a role in addition similar to that played by *one* in multiplication.

It seems to me that per cent is often "built up" into a concept much deeper and "tougher" appearing than is really the case. Once the definition $r\%=\frac{r}{100}$ is established and it is accepted that the two expressions are completely interchangeable with each other and with their decimal equivalent, I feel that it is unwise and

certainly unnecessary to stress the learning of "type" problems. A very nice application of multiplication by *one*, in the form of 100%, occurs often in such problems. If, for example, we are required to determine what per cent of 4 is 32, an elegant beginning is made when we write the obvious truth, $4 = 100\%$ of 4. Then the familiar argument, four apples cost twenty cents, one apple costs five cents, three apples cost fifteen cents, leads immediately to $32 = 800\%$ of 4.

As I have indicated previously I am one of those who believe that arithmetic should lead *naturally* to algebra. Consider, for instance, the following typical problem in seventh-grade arithmetic: The prices of dresses have been reduced thirty per cent. If a dress is now selling for thirty-five dollars, what was its original price? If we call the original price P, or any other symbol which we would find convenient to use, and proceed to operate with P just as if we knew its value, we obtain at once the equality $\frac{70}{100} \times P = 35$. The actual value of P is easily deduced by purely arithmetical reasoning. Again, the principal advantage gained is in freeing the pupil from "type" procedures which normally must be predetermined for this problem and for others where the given information has been varied.

I realize that anyone suggesting the use of a letter to represent an unknown quantity in arithmetic problems is likely to be met with cries of anguish from many people who seemingly possess a congenital predisposition against such an innovation. I hope the following example may help to answer some of the objections. Let us evaluate the fraction which is equal to the repeating decimal $.232323 \cdots$. We may define $.\dot{2}\dot{3} = .232323 \cdots$ and from then on, whether we like it or not, the symbol $.\dot{2}\dot{3}$ plays exactly the role of the unknown quantity and could be represented equally well by x or n or ? or #. Multiplying this quantity by 100 and subtracting the original repeating decimal from it gives us $99(.\dot{2}\dot{3}) = 23$ and the rational form is obtainable immediately. The question to use or not to use a letter in such a case should not be such an earth-shaking decision to make. The person who does not wish to use a letter in arithmetic cannot argue that it will "complicate" the procedure.

Transferring our attention to more abstract considerations, I feel that elementary teachers should be well aware that the converse of a theorem is not always true. A trivial example will suffice: all squares of whole numbers are whole numbers but not all whole numbers are the squares of whole numbers. I am reminded of the passage in *Alice in Wonderland* in which the mathematician Lewis Carroll has his characters speak as follows: "Then you should say what you mean," the March Hare went on. "I do," Alice hastily replied. "At least I mean what I say—that's the same thing you know." "Not the same thing a bit," said the Hatter. "Why you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see.' " "You might just as well say," added the March Hare, "that 'I like what I get' is the same thing as 'I get what I like.' " "You might just as well say," added the Dormouse, which seemed to be talking in its sleep, "that 'I breathe when I sleep' is the same thing as 'I sleep when I breathe.' "

Similarly, an untrue assumption can lead to any conclusion and hence the truth of a conclusion does not establish the truth or falsity of the original assumption. Perhaps the following will illustrate: If it is assumed that $3 = 1$, then we may obtain $4 = 2$, $2 = 1$, $7 = 3$, etc., as well as $4 = 4$.

Advanced mathematics is generally carried on by proving theorems based on definitions, axioms, postulates and hypotheses. Even though a theorem may hold in an infinity of individual cases it cannot be true in general if a single set of circumstances exists for which the theorem is not satisfied. If such a counter-example can

be found, this is the simplest and most efficient way of disproving the theorem. For instance, one might conjecture that the fact that one may cite an infinity of examples of the following type: $\frac{12}{24} \neq \frac{1}{4}$, $\frac{17}{37} \neq \frac{1}{3}$, $\frac{23}{35} \neq \frac{2}{5}$, etc., might lead to a general theorem of that nature. But we know that $\frac{16}{64} = \frac{1}{4}$ and there are other similar cases, e.g., $\frac{49}{98} = \frac{4}{8}$ and $\frac{19}{95} = \frac{1}{5}$, to disprove the proposition.

Another bit of logic which can be helpful is the law of the contra-positive, viz., the fact that P implies Q is equivalent to *not* Q implies *not* P. Very often its use may serve to emphasize the inaccuracy of a general statement. The following trivial example will illustrate the equivalence: If a whole number is divisible by 4, then it is divisible by 2; and equivalently, if a whole number is not divisible by 2, then it is not divisible by 4.

In this discussion I have attempted to point out some of the many consistencies which present themselves repeatedly in arithmetic and which can be used to great advantage in clarifying the subject, viz., the symmetry of equality, the commutativity of multiplication, the special properties of *zero* and *one*, the advantages of using parentheses and simple algebraic techniques and some elementary principles in logic. Although we may not make tremendous use of such mathematics in elementary arithmetic, I believe some acquaintance with the elegance inherent in these fundamentals will be reflected in the enthusiasm and understanding with which a teacher presents arithmetic to a class.

Have you read?

Kline, Morris. "Projective Geometry," *Scientific American*, vol. 192, January 1955, pp. 80–86.

Here is an article no mathematics teacher or student should fail to read. From the cover painting to the last line in the article it is exciting reading. From the title one might think it is an advanced treatise but the author has done an excellent job of showing how projective geometry is an integral part of all mathematics.

It is of interest to note how the Renaissance painters created projection and section to represent three dimensions on a two-dimensional plane. The mathematicians then became intrigued and raised the question of elements common to all sections of the same projection. For example, what happens to all triangles, quadrilaterals, polygons and circles? You will be interested in the discussion of Desargues' contribution and its rejection by mathematicians of his time; the contributions of Pascal and how these were almost lost. The author gives an excellent discussion of duality and how one dualizes. This gives an insight into the process of "creating" mathematics.

You will want to see how projective geometry is related to Euclidean geometry and the impetus projective geometry gave to topology. The article is well summarized by the statement, "No branch of mathematics competes with projective geometry in originality of ideas, coordination of intuition in discovery and rigor in proof, purity of thought, logical finish, elegance of proofs and comprehensiveness of concepts."

Tinker, Miles A. "Readability of Mathematical Tables," *Journal of Applied Psychology*, vol. 38, December 1954, pp. 436–442.

The table as an arrangement for the presentation of data and information is very important today. Many of our students fail to consider the table itself but think of it only for the information it gives. This article summarizes a study of tables and their use. It answers such questions as: What is the optimum number of columns in a table? In pages of tables for finding squares, square roots, etc., should the pages be numbered or should the page be labeled with the number sought? What is the best size of type? Does the number style make any difference? What kind of paper and size of page is best? How about the spacing of the numbers?

This study is one of the few done in this area and the information given here can be used with your classes to motivate and make tables more interesting to the student.—Philip Peak, *Indiana University, Bloomington, Indiana*

Mathematics used in courses of various departments in a university

OTHO M. RASMUSSEN, *University of Denver, Denver, Colorado.*
A study pointing out some of the mathematical needs of university students and the shortcomings of present mathematics courses.

THERE IS a growing tendency to introduce, in college mathematics, beginning courses which are of a type other than the traditional college algebra and trigonometry. Broader courses have been designed to contribute toward the achievement of objectives of general education at the college level. The content of these courses varies greatly according to the level of difficulty and the purpose the course is designed to serve.

One of the purposes of almost all of the newer courses is to develop the mathematical understandings and skills that students need in other fields of study in college. A few reports have been made of studies to determine the mathematical needs of students in certain specialized fields, but about two years ago the writer was unable to find a report of any systematic investigation that had been made to determine what the content of a first-year general mathematics course should be in order to serve the needs of students generally throughout a university.

The present article describes a study made to determine those mathematical skills and concepts which are desirable as preparation for study in courses for university students not majoring in mathematics or the physical sciences. This study was conducted at the University of Kansas in the year 1951–1952.

The first step was to prepare a list of courses throughout the university which were considered to involve the use of some mathematics. Eliminated from consideration were: all courses in the mathematics department; all professional engineering courses; those courses in the various physical science departments beyond the elementary course; and advanced courses of a specialized or technical nature in other departments. A beginning was made by including all other courses in the university which had any mathematics prerequisite. Textbooks, manuals, and other materials used in these and other courses were examined for the use of mathematical concepts. As a result, some courses were added to the list. Then the chairmen of departments in the liberal arts college and the deans of certain of the professional schools were consulted to obtain information about courses in their fields in which mathematics was being used. On the basis of information gained in these conferences, several courses were removed from the list and others were added. The chairmen of the departments of botany, entomology, geology, philosophy, physiology, political science, sociology and anthropology, and social work all stated that no mathematics was used in the elementary courses in their departments. The chairman of the department of social work expressed the opinion that perhaps some elementary mathematics should be provided for graduate students in that department.

The compilation resulted in a list of

TABLE 1.—MATHEMATICS TOPICS FOUND IN COURSE TEXTBOOKS

Mathematics topics	Professional schools				Liberal arts departments							
	Business	Education	Journalism	Pharmacy	Astronomy	Bacteriology	Chemistry	Geography	Home economics	Psychology	Phys. science	Physics
1. Fund. processes of arith.	X	X	X	X	X		X					X
2. Percentage	X	X	X	X		X	X	X	X			X
3. Geometry, simple	X				X		X					
4. Ratio, proportion, and variation	X	X		X	X		X	X	X	X	X	X
5. Linear equations	X	X	X	X		X	X	X		X	X	X
6. Simult. linear equations	X											
7. Quadratic equations							X					X
8. Powers, roots, and radicals		X		X	X	X	X	X		X	X	X
9. Logarithms	X	X		X	X	X	X					X
10. Approximate computation	X	X					X					X
11. Trigonometry				X	X							X
12. Graphs and tables	X				X		X		X	X	X	X
13. Permutations, combinations, and probability	X						X			X		
14. Statistics	X	X		X	X		X	X	X	X	X	X
15. Vectors	X										X	X
16. Conic sections	X				X							
17. Misc. symbols	X						X			X		X
18. Misc. terms	X											X

courses in the following schools and departments:

Professional schools	*Liberal arts departments*	
Business	Astronomy	Home Economics
Education	Bacteriology	Psychology
Journalism	Chemistry	Physical Science
Pharmacy	Geography	Physics

A list of required text materials for each course in the list was obtained from the chairman of the department or the instructor in each course. A careful examination was made of the textbooks of the courses considered to learn what mathematical skills and concepts were involved, and a tentative list of topics was prepared. This list was revised as it was deemed advisable when the textbooks were examined, and tabulations made of the occurrence of these topics in the texts. As each textbook was examined, each mathematical topic that was found to be involved was tabulated but once regardless of the number of times it appeared in the text. The results of the analysis of the textbooks were summarized by departments and by schools. The resulting list included those topics shown in Table 1. For any given topic it can be seen which schools and departments make use of that topic in their textbooks. For example, logarithms are used in the textbooks of elementary courses in the schools of business, education, and pharmacy, and in the departments of astronomy, bacteriology, chemistry, and physics.

The following items are included under the various topics of Table 1 and Table 2:

1. *Fundamental processes of arithmetic:* The four fundamental operations applied to

whole numbers, fractions, and decimals; extraction of square roots.

2. *Percentage:* Understanding of the concepts and terminology; computation involving the various "cases."
3. *Simple geometry:* Similar triangles; concept of tangent to a curve; asymptote; interpretation of three-dimensional line drawings.
4. *Ratio, proportion and variation:* Arithmetic and algebraic examples.
5. *Linear equations:* Solution of linear equations in one unknown; substitution in formulas and the solving of formulas for one unknown in terms of others; the formulation of equations.
6. *Simultaneous linear equations:* Solution of pairs of linear equations in two unknowns.
7. *Quadratic equations:* Solution of quadratic equations in one variable with numerical coefficients.
8. *Powers, roots, and radicals (other than the computations included in 1 and 9):* The meaning and use of powers and roots; expressing numbers as positive and negative powers of 10.
9. *Logarithms:* Meaning of a logarithm; properties of logarithms; computations involving multiplication, division, raising to a power, and extraction of roots.
10. *Approximate computation:* Significant digits.
11. *Trigonometry:* Definitions of sine, cosine, and tangent of acute angles; solution of right triangles; inverse sine; sine curve; radian measure.
12. *Graphs and tables (other than statistical):* Understanding functional relationships expressed in graphical and tabular form; constructing tables and graphs to express functional relationships.
13. *Permutations; combinations; and probability, including the normal probability curve:* Applications of the most elementary principles.
14. *Statistics:* Arrangement and interpretation of data; measures of central tendency; measures of dispersion; correlation; construction and interpretation of statistical graphs.
15. *Vectors:* Addition of vectors.
16. *Conic sections:* Simple properties of the ellipse, including eccentricity; meaning of the expression, "rectangular hyperbola."
17. *Symbols:* Meaning of $\sum$, ΔX, $<$, $>$, and $|a|$.
18. *Miscellaneous terms:* Discontinuity; slope of a curve; angle of inclination of a line.
19. *Fundamental operations of algebra:* The four fundamental operations; functional notation; converting word problems to formulas. (There obviously is some overlapping between this item and the item of Linear equations.)
20. *Binomial theorem.*
21. *Progressions:* Arithmetic and geometric.
22. *Slide rule:* Use in multiplication, division, and extraction of square roots.

In individual conferences with the instructors of the courses under consideration the instructors were informed of the purpose of the study by the writer after he had made the analysis of the textbooks used in their courses. They were asked to fill out a questionnaire concerning the use of mathematics in their courses. Within a few days after the questionnaire had been given to each instructor the author met with him to discuss his answers to the questions. In those cases where answers appeared to be incomplete or not clear, the instructor was asked to furnish additional information or to explain his answer. In each case in which the instructor did not give all of the mathematical topics which had been found in the analysis of the textbook which he used, he was asked about the omitted items.

A tabulation was made by departments and by schools to show the mathematical topics which the instructors indicated were used in their courses. These topics appear in Table 2 and differ in some respects from these resulting from the analysis of textbooks.

There were a number of cases in which a topic appeared in the textbook of a course but in which the instructor of the course indicated that the material was presented in such a way that the item was not involved. The principle of addition of vectors, for example, was found in the textbooks of three courses, whereas instructors stated that this principle was not used in the courses. The terms "discontinuity," "slope of a curve," and "angle of inclination of a line," which were found in the textbook of an economics course, were not used in the course. Logarithms appeared more frequently in the textbooks than in the replies of the instructors.

Some mathematical principles in addition to those found in the analysis of textbooks were mentioned in the interviews. Instructors indicated several applications of simple geometry which had not been found in the textbooks of the courses. Under trigonometry was added the law of

TABLE 2.—MATHEMATICS TOPICS FOUND IN INTERVIEWS

Mathematics topics	Professional schools: Business	Education	Journalism	Pharmacy	Liberal arts departments: Astronomy	Bacteriology	Chemistry	Geography	Home economics	Psychology	Phys. science	Physics
1. Fundamental processes of arith.	X	X	X	X	X	X	X	X	X		X	X
2. Percentage	X	X	X	X		X	X	X	X			
3. Geometry, simple	X				X		X	X			X	X
4. Ratio, proportion, and variation	X	X	X	X	X	X	X	X	X	X	X	X
5. Linear equations	X		X	X			X	X		X		X
6. Simult. linear equations	X			X								X
7. Quadratic equations	X						X					X
8. Powers, roots, and radicals	X				X	X	X	X			X	X
9. Logarithms	X			X			X			X		X
10. Approximate computation	X						X					X
11. Trigonometry				X	X			X				X
12. Graphs and tables	X	X		X	X		X	X		X	X	X
13. Permutations, combinations, and probability	X	X								X		
14. Statistics	X	X	X	X	X		X	X	X	X	X	X
16. Conic sections					X							
17. Misc. symbols	X						X			X		X
19. Fundamental operations of algebra	X			X	X						X	X
20. Binomial theorem	X											
21. Progressions	X							X				
22. Slide rule	X						X	X				

sines, law of cosines, and the identities

$$\tan A = \frac{\sin A}{\cos A}, \quad \text{and} \quad \sin^2 A + \cos^2 A = 1.$$

The process of solving certain quadratics by disregarding the linear term was added to the topic of approximate computation.

In tabulating the results of the interviews, a topic was included for a school or a department even if it were mentioned only by a single instructor. No attempt was made to weight a topic according to the emphasis given it by instructors.

The following conclusions appear to be warranted concerning the mathematical needs of general students at the University of Kansas:

1. For a vast majority of the students who are not majors in mathematics or physical science, the mathematics needed for successful completion of their undergraduate program is quite elementary.

2. An understanding of the fundamental operations of arithmetic is necessary in a large number of university courses.

3. A majority of students do not possess sufficient arithmetical maturity to enable them to gain maximum benefit from a large number of university courses.

4. Much of the material normally presented in traditional college algebra and trigonometry courses is not used in courses outside of those in the mathematics and physical science departments and is not

used in the elementary courses in astronomy, chemistry, and physics. Those parts of trigonometry which involve secants, cosecants, and cotangents and many of the more advanced topics are not used by students who are not majoring in mathematics or physical science. Very little of the usual algebraic material concerning simultaneous linear equations and quadratic equations and none of the work on theory of equations is used in other courses by these students.

5. There is a definite need in many courses in departments throughout the university of understanding of elementary statistics.

6. In a considerable number of departments more mathematics is needed for advanced work than is needed for elementary courses. An understanding of elementary statistics is necessary for advanced work in most departments.

In this study no consideration is given the question of whether the present courses throughout the university are properly organized and taught to achieve valid educational objectives. This study, however, in no way implies that the status quo should be maintained. As improvements are made in the entire university program, instruction in mathematics should continue to contribute to the effectiveness of the entire program.

This investigation is restricted to a single university. Although in many respects this institution appears to be typical of a large number of American universities, the possibility exists that certain of the findings resulted from local factors and caution must therefore be used in applying the results of this study to courses on other campuses.

Because this study involved the use of questionnaire and interview techniques the results are subject to the limitations that are inherent in these techniques. Although great care has been exercised in using these techniques, it is conceivable that in some cases the responses of instructors were influenced by preconceived notions with respect to the importance of mathematics.

Bibliography

Allendoerfer, C. B. "Mathematics for Liberal Arts Students," *The American Mathematical Monthly*, 54: 573–577, December, 1947.

Boyer, Lee Emerson. *College General Mathematics for Prospective Secondary School Teachers*, The Pennsylvania State College Studies in Education, Number 17. State College, Penn.: School of Education, Pennsylvania State College, 1939. 106 pp.

Brink, R. W. "A Course in Mathematics for the Purpose of General Education," *Journal of General Education*, 1: 279–286, July 1947.

Brown, Kenneth E. "The Content of a Course in General Mathematics—Teacher's Opinions," The Mathematics Teacher, 43: 23–30, January 1950.

Georges, J. S. "Mathematics in the Junior College," *School Science and Mathematics*, 37: 302–316, March 1937.

Griffin, F. L. "An Experiment in Correlating Freshman Mathematics," *The American Mathematical Monthly*, 22: 325–330, December 1915.

"Guidance Report of the Commission on Post-War Plans," The Mathematics Teacher, 40: 315–339, November 1947.

Hannelly, Robert Jeffrey. "Mathematics in the Junior College," Unpublished Doctor's thesis, University of Colorado, Boulder, 1939.

Hannon, Herbert H. "An Investigation to Determine the Topical Content for a Course in Mathematics for General Education at Western Michigan College of Education, Kalamazoo, Michigan," Unpublished Doctor's field study, Colorado State College of Education, Greeley, 1951.

Moore, E. H. "On the Foundations of Mathematics," *Bulletin of the American Mathematical Society*, 9: 402–424, May 1903.

Newsom, C. V. "A Course in College Mathematics for a Program of General Education," The Mathematics Teacher, 42: 19–24, January 1949.

Northrup, E. P. "Mathematics in a Liberal Education," *The American Mathematical Monthly*, 52: 132–137, March 1945.

Nunn, T. Percy. *The Teaching of Algebra*. New York: Longmans, Green and Co., 1914. 616 pp.

Osner, Henry. "Junior-College Survey Course in Mathematics," *California Journal of Secondary Education*, 22: 468–472, December 1947.

Report by the National Committee of Mathematics Requirements under the auspices of the Mathematical Association of America,

Inc., *The Reorganization of Mathematics in Secondary Education*, 1923. 625 pp.

Report of the Harvard Committee of the Objectives of a General Education in a Free Society, *General Education in a Free Society*, Cambridge, Massachusetts: Harvard University Press, 1945. 267 pp.

RICHTMEYER, CLEON C. "Functional Mathematics Needs of Teachers," *Journal of Experimental Education*, 6: 396–398, June 1938.

SCHORLING, RALEIGH. "A Program for Improving the Teaching of Science and Mathematics," *The American Mathematical Monthly*, 55: 221–237, April 1948.

"The Second Report of the Commission on Post-War Plans," THE MATHEMATICS TEACHER, 38: 195–221, May 1945.

TRIMBLE, H. C. "Basic Mathematics for General Education," *Journal of General Education*, 3: 49–52, October 1948.

WOLFF, GEORG. "Mathematics as Related to Other Great Fields of Knowledge," *Eleventh Yearbook of the National Council of Teachers of Mathematics*. New York: Bureau of Publications, Teachers College, Columbia University, 1936. Pp. 207–244.

YOUNG, J. W. "The Organization of College Courses in Mathematics for Freshmen," *The American Mathematical Monthly*, 30: 6–14, January 1923.

ZANT, J. H. "A Course in Freshman Mathematics," *The American Mathematical Monthly*, 48: 246–247, April 1941.

Exactly what is a nautical mile?

Generations of schoolboys through the years have pondered the distinction between statute miles and nautical miles. The English statute mile, used in the United States and in British countries, equals 320 rods, or 1760 yards, or 5280 feet. The nautical mile, or sea mile, is the length of one minute of a great circle of the earth. It is now also the air mile.

These definitions seem straightforward enough. But the earth is not a perfect sphere. So which great circle of the earth should furnish the official minute's length? Among many possible values the British have traditionally used 6080 feet, the French 6076.2 feet, and the United States 6080.20 feet.

To encourage uniformity, the International Hydrographic Bureau proposed in 1929 that 1852 meters (6076.103 United States feet) be accepted internationally. And most of the nations of the world did. There were, however, three notable exceptions, namely, the United States, Great Britain, and Russia.

More recently, however, the matter revived. Why should the United States, a leader in commerce, hold to its own unique nautical mile? For a time opposition to a change came from the Civil Aeronautics Administration. It was just in the midst of directing a transfer nationally from the statute mile to the nautical mile. Two simultaneous changes, it feared, would confuse people.

Once the transition from the statute mile to the nautical mile occurred, though, the next step was to alter the nautical to conform to the international nautical mile. This was done in 1954 when the Secretaries of Defense and Commerce signed an agreement effective July 1.

By October 1 United States aviation was flying in terms of the international nautical mile. Even though the present unit is four feet shorter than the former one, no major effects can be noticed. Everything is proceeding as smoothly as before. Charts, maps, instruments, and navigational procedures continue to be effective without any alteration whatsoever. The nautical mile has no applications wherein a change of 4 parts in 6080 renders it too unprecise for use. The change was too small to be in any way disruptive.

"As regards motivating the formal proof itself, I wish to point out that understanding each step of a proof is no indication of understanding the proof looked at as a whole. Similarly, if one understands each theorem it does not mean one understands the subject matter as a whole. We should not lose sight of the *Gestalt* by looking too much at details. Learning mathematics by understanding in turn each sentence in a formal style of presentation can become a cook-book style of learning. Hence I would stress the 'idea' of a proof as much as the formal proof itself, and the 'idea' should precede the formal proof. However, one should emphasize that the 'idea' falls short of the requirements of a formal proof and these shortcomings should be explicitly pointed out. When the formal proof is completed one should tie in loose ends by checking the places where the hypotheses were needed in the proof. If time is pressing, as it always is, I would much prefer to see the formal proof omitted rather than omitting the motivating remarks and/or the 'idea' of the proof (especially since the formal proof can be found in texts while the motivation is not often enough in print). However, if this program is too time-consuming for the lecturer, part of it could be assigned explicitly as a problem for the student."—*Howard Raiffa*, "*Mathematics for the Social Scientist*," The American Mathematical Monthly, *October 1954*

Involution operated geometrically

JUAN E. SORNITO, *Central Philippine University, Iloilo City, Philippines.*

High school students of plane geometry may be interested in the central theme of this paper as a generalization of the usual problem of finding a line equal to the square of a given line.

THE SUBJECT of representing the fundamental operations by the methods of plane geometry offers a fertile field for investigation. The addition, subtraction, multiplication and division of straight lines are familiar to students of mathematics. The following are two methods of representing involution in terms of straight lines, i.e., the determination of a straight line equal to a power of a given straight line.

Given: The line a.

Problem: To determine the length of the line equal to a^n.

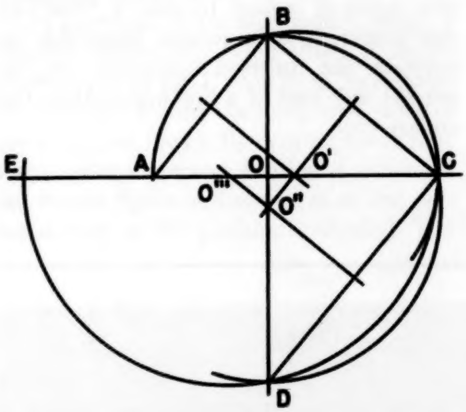


Figure 1

FIRST METHOD

In Figure 1 let AO equal the unit length a. At O erect a perpendicular to AO. Measure OB equal to a, the given line. Draw AB and erect its perpendicular bisector. Let this perpendicular intersect AO prolonged, at O'. With O' as center describe an arc passing through A and B. Let this arc intersect AO, prolonged at C. Draw BC and erect its perpendicular bisector. Let this perpendicular intersect BO prolonged, at O''. With O'' as center describe an arc passing through B and C. Let this arc intersect BO at D. By the same process O''' and E may be determined. This process can be continued indefinitely.

In the figure

$$OC=a^2;\ OD=a^3;\ OE=a^4 \cdots \text{etc.}$$

Proof:

In the right triangle ABC

$$AO:OB=OB:OC$$

substituting 1 for AO and a for OB

$$1:a=a:OC$$
$$OC=a^2.$$

In the right triangle BCD

$$OB:OC=OC:OD$$

substituting a for OB and a^2 for OC

$$a:a^2=a^2:OD$$
$$a\cdot OD=a^4$$
$$OD=a^3.$$

In the right triangle CDE

$$OC:OD=OD:OE$$

substituting a^2 for OC and a^3 for OD

$$a^2:a^3=a^3:OE$$
$$a^2\cdot OE=a^6$$
$$OE=a^4$$

This process may be continued to obtain a^n.

SECOND METHOD

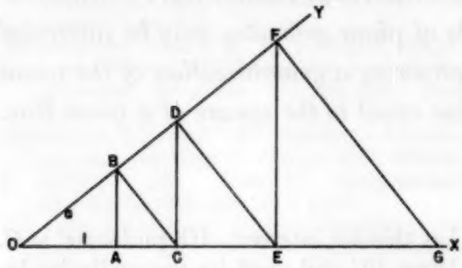


Figure 2

In Figure 2 let OA on the line OX equal the unit length. At A erect a perpendicular to OX. By the use of a compass measure OB equal to a, the given line. Through B draw OY. Construct BC perpendicular to OY; CD perpendicular to OX; DE perpendicular to OY; EF perpendicular to $OX \cdots$ etc.

Then $OC=a^2$; $OD=a^3$; $OE=a^4 \cdots$ etc.

Proof:

In the right triangle OBC, AB is perpendicular to OC

$$OA:OB=OB:OC$$

substituting 1 for OA and a for OB

$$1:a=a:OC$$

$$OC=a^2.$$

In the right triangle OCD, BC is perpendicular to OB

$$OB:OC=OC:OD$$

substituting a for OB and a^2 for OC

$$a:a^2=a^2:OD$$

$$a\cdot OD=a^4$$

$$OD=a^3.$$

In the right triangle ODE, DC is perpendicular to OE

$$OC:OD=OD:OE$$

substituting a^2 for OC and a^3 for OD

$$a^2:a^3=a^3:OE$$

$$a^2\cdot OE=a^6$$

$$OE=a^4.$$

It will be observed from the foregoing that while it is easy to raise a "line" to any power, it is in general impossible to perform the reversed operation; i.e., to extract any root of a line higher than the square root.

"Given certain factors, and a sound brain should always evolve the same fixed product with the certainty of Babbage's calculating machine.

"What a satire, by the way, is that machine on the mere mathematician! A Frankenstein-monster, a thing without brains and without heart, too stupid to make a blunder; which turns out results like a cornsheller, and never grows any wiser or better, though it grind a thousand bushels of them!

"I have an immense respect for a man of talents plus 'the mathematics.' But the calculating power alone should seem to be the least human of qualities, and to have the smallest amount of reason in it; since a machine can be made to do the work of three or four calculators, and better than any one of them. Sometimes I have been troubled that I had not a deeper intuitive apprehension of the relations of numbers. But the triumph of the ciphering hand-organ has consoled me. I always fancy I can hear the wheels clicking in a calculator's brain. The power of dealing with numbers is a kind of 'detached' lever arrangement, which may be put into a mighty poor watch. I suppose it is about as common as the power of moving the ears voluntarily, which is a moderately rare endowment."—*Oliver Wendell Holmes,* The Autocrat of the Breakfast-Table

Edited by Emil J. Berger, Monroe High School, St. Paul, Minnesota

A model for visualizing the formula for the area of a circle

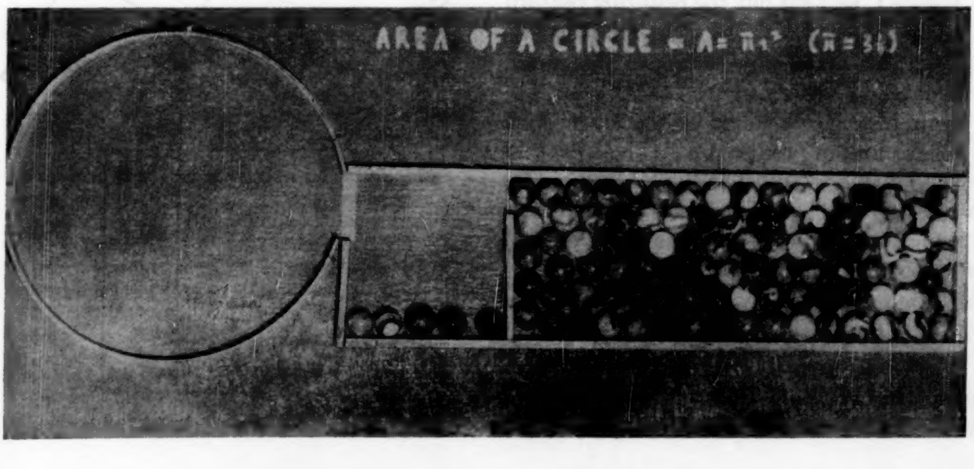
by Clarence Olander, St. Louis Park, Minnesota

Developing models and materials for visualization is important at all levels of instruction, but especially so at the junior high level. In this brief note an attempt has been made to describe the construction of a device which, it is hoped, readers may find useful in helping their pupils develop an understanding of the fact that the area of a circle is approximately $3\frac{1}{7}$ times the square of the radius.

To construct a suitable model, select a wooden embroidery hoop about 7″ in diameter, cut an arc 1″ long from its circumference, and mount the hoop on a piece of plywood (see illus.). By making use of four narrow strips of wood, fashion a rectangular shaped figure and mount it on the plywood sheet in the position indicated. The inside dimensions of this rectangle should be respectively equal to and four times the length of the radius of the hoop. The side of the rectangle next to the hoop must have an opening which corresponds with the opening in the hoop. Finally, mount a small piece of wood (one whose length is $\frac{3}{8}$″ shorter than the radius of the hoop) inside the rectangle so that the area of the rectangle is divided in the ratio 1:3.

By filling the circle completely with marbles and allowing them to pass through the opening into the rectangle the fact that the area of a circle is approximately $3\frac{1}{7}$ times the square of its radius becomes plausible.

A device like this should be available so that every class member may have an



opportunity to manipulate it individually. In this informal way, pupils can, by making actual measurements, satisfy themselves that the formula $A = \pi r^2$ is reasonable. The model is easy to construct and would be a worth-while pupil project.

A model for visualizing the Pythagorean theorem

by Emil J. Berger

Most teachers of plane geometry attempt to help their students gain an understanding and appreciation of the Pythagorean theorem by having them develop a proof of it. This effort is often concluded with an algebraic proof because it is claimed that proofs which involve comparison of areas are too difficult for tenth-grade students. However, this writer is of the opinion that students will gain significantly both in understanding and appreciation if they study some proof of the last named type. Since many textbooks include the proof which is sometimes ascribed to Euclid, it seems appropriate for this department to suggest a model for this particular proof.

To construct a model procure a piece of masonite $\frac{1}{4}'' \times 24'' \times 24''$, make a drawing on it similar to the diagram proposed in Figure 1, and cut out the resulting pattern as one piece. Paint the right triangle yellow, and make the three squares gray. Lines and letters appearing on the diagram may be stencilled on the model with black paint or produced with black Scotch-type tape $\frac{1}{4}''$ wide. The little black circles carrying "+" signs indicate the points at which dowel pegs $\frac{1}{4}''$ in diameter and $\frac{5}{8}''$ long should be located.

Figure 2 indicates the shapes and relative sizes of the two pairs of congruent triangles which are normally introduced in Euclid's proof by means of auxiliary construction lines. All four triangles should be cut from semi-transparent plastic $\frac{1}{8}''$ thick. Their purpose for this model is best served if each pair is cut from a different shade of material (e.g., red and green respectively). Experience with this model has shown that it is not necessary that these triangles be lettered. The lettering which appears in Figure 2 has been introduced to facilitate reference for the reader. The little circles which appear in each triangle are 5/16″ holes. Each must be centered so that when all four triangles are correctly located on the model (Figure 3) the holes correspond with the dowel pins set in the masonite.

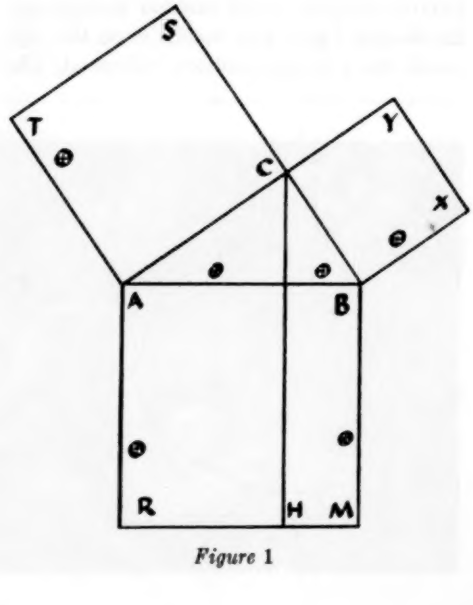


Figure 1

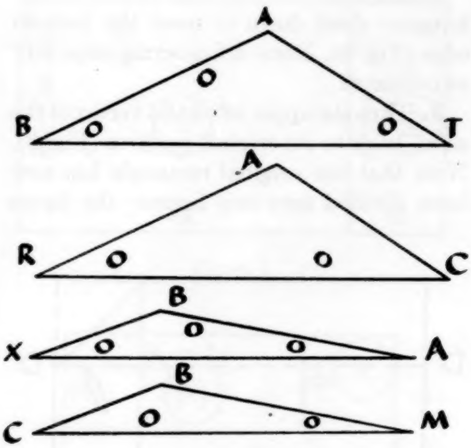


Figure 2

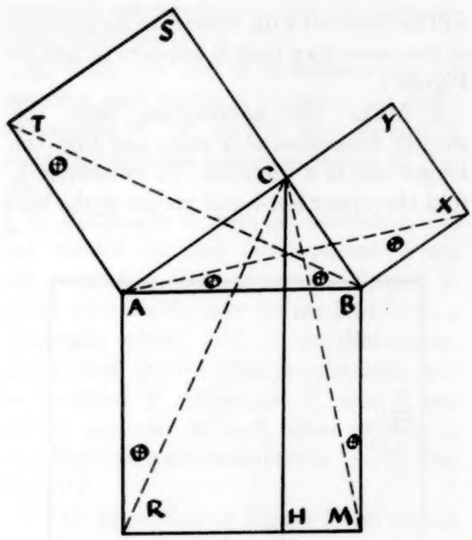


Figure 3

The difficulty that students normally encounter with the proof suggested by Figure 3 is that they are unable to bring order to the maze of auxiliary construction lines that are needed. Using the model makes it possible to isolate the particular pair of triangles under consideration for the moment, and to remove them from the scene when another pair is to be considered. Once the proper pairs of figures have been compared and related the composite figure (Figure 3) usually takes on real meaning for the student. The informal approach suggested in this article should add considerably to students' comprehension and appreciation of the proof. It hardly seems necessary to mention that this model can be used as an aid to help the student discover the proof for himself.

Teaching aid for developing (a + b) (a − b)

by Carl Uth, Benton Harbor High School, Benton Harbor, Michigan

Editor's Note: The concept-developing technique included in this contribution is as clever as any that the postman has ever dropped into this department editor's mailbox. So that the reader may experience the "feel" of the development as presented it is suggested that he supply himself with a sheet of $8\frac{1}{2}'' \times 11''$ typing paper, and observe the directions in the order listed. Doing this will help the reader understand what the experience may be like for his students. The plan outlined below should prove to be a profitable classroom activity for ninth-grade algebra classes.

The purpose of the procedure outlined in this article is to help students develop an understanding of the concept embodied in the following relation:

$$(a+b)(a-b)=a^2-b^2$$

The approach is geometric and depends on the area concept. The only materials needed for this development are a rectangular sheet of typing paper, a pencil, a straightedge, and a scissors. The plan of the development is contained in the following sequence of steps:

1. Place the sheet of typing paper on a working table or desk so that the reader

will be oriented with respect to its position in the same way that it appears to him in Figure 1.

2. Make the assumption that the shorter dimension is a units and that the longer one is $a+b$ units. To determine b, fold the upper left-hand vertex of the rectangular sheet down to meet the bottom edge (Fig. 2). Draw a line along edge EE′ as indicated.

3. Turn the upper left-hand vertex of the sheet back to its original position (Fig. 3). Note that the original rectangle has now been divided into two figures; the figure

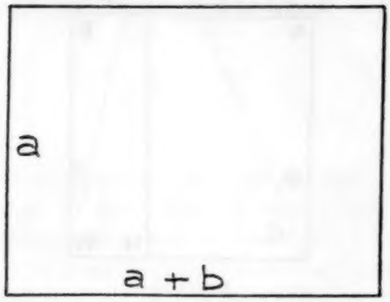


Figure 1

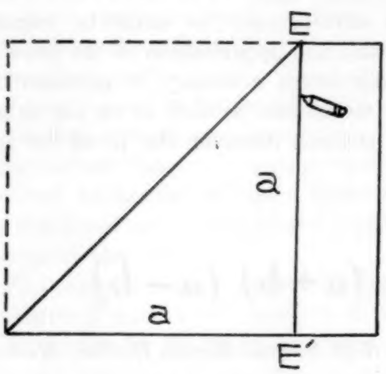


Figure 2

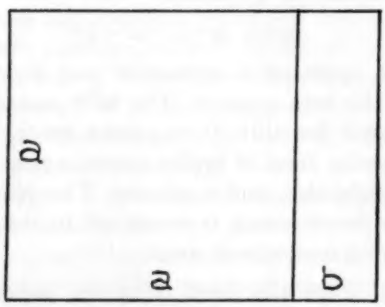


Figure 3

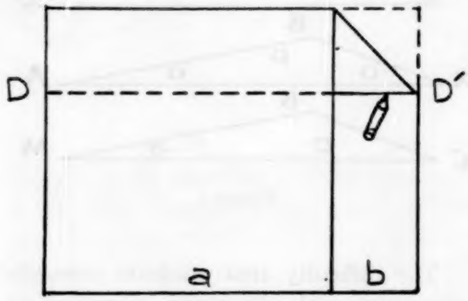


Figure 4

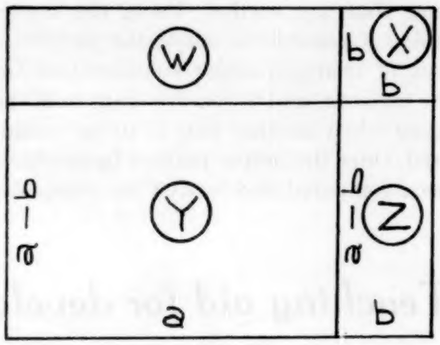


Figure 5

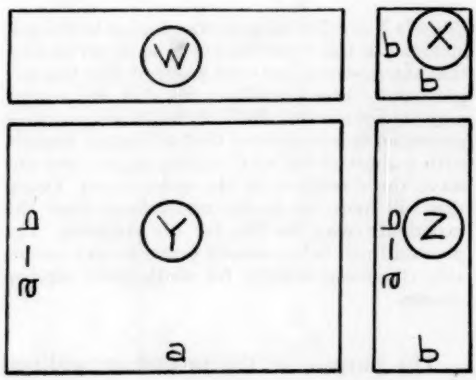


Figure 6

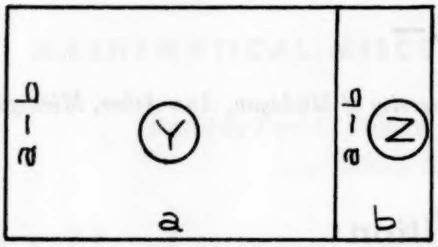


Figure 7

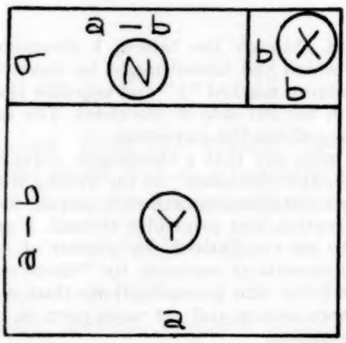


Figure 8

on the left is a square which is a units on a side, and the figure on the right is a rectangle whose dimensions are b and a.[1] Thus the magnitude of b is determinate.

4. Fold the upper right-hand vertex down as shown in Figure 4, and with the aid of a straightedge, draw line DD′.

5. Label the various edges and lines with their dimensions as indicated in Figure 5, and, as a simple expedient for clarifying references, "key" the four figures formed with the letters W, X, Y, and Z. Check to insure that the dimensions recorded are consistent with the assumptions made in step 2.

6. With the aid of a scissors (or by creasing and tearing) separate the three rectangles W, Y, and Z, and square X into four separate pieces as shown in Figure 6.

7. Rectangle W may be discarded; it is not needed for this development. From the remaining three pieces X, Y, and Z, select two which may be used to form a rectangle which will have dimensions $(a+b)$ and $(a-b)$. This requirement will be satisfied if rectangles Y and Z are placed adjacent to each other in such a way that their common side is $(a-b)$. See Figure 7.

8. An inspection of Figure 7 will reveal that the area of Y is $a(a-b)$ or a^2-ab, and that the area of Z is $b(a-b)$ or $ab-b^2$.

9. Thus $(a+b)\ (a-b)=a^2-b^2$.

The device described in the foregoing steps may also be used to provide a geometric interpretation of the fact that the factors of a^2-b^2 are $(a+b)\ (a-b)$. Arrange rectangles Y and Z and square X as illustrated in Figure 8. The area of the three pieces is a^2 certainly. By removing (subtracting) b^2 (keyed X), and again arranging rectangles Y and Z as in Figure 7, the desired result is immediately apparent.

Anyone who has a learning aid which he would like to share with fellow teachers is invited to send this department a description and drawing for publication. Or, if that seems too time-consuming, simply pack up the device and mail it. We will be glad to originate the necessary drawings and write an appropriate description. All devices submitted will be returned as soon as possible. Send all communications to Emil J. Berger, Monroe High School, St. Paul, Minnesota.

[1] To facilitate communication the writer has adopted the convention of referring to rectangles with unequal adjacent sides as "rectangles," and of reserving the word "square" for reference to rectangles with equal adjacent sides.

• HISTORICALLY SPEAKING,—

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Tangible arithmetic I: a correction and an addition

by Robert R. Wright, Sampson Air Force Base, Geneva, New York

We are particularly delighted to have Robert R. Wright point out in the following letter how historical materials may interest, motivate, and, most important of all, lead to a "discovery."—*P.S.J.*

31 December 1954

Phillip S. Jones
University of Michigan
Ann Arbor, Michigan

Dear Mr. Jones:

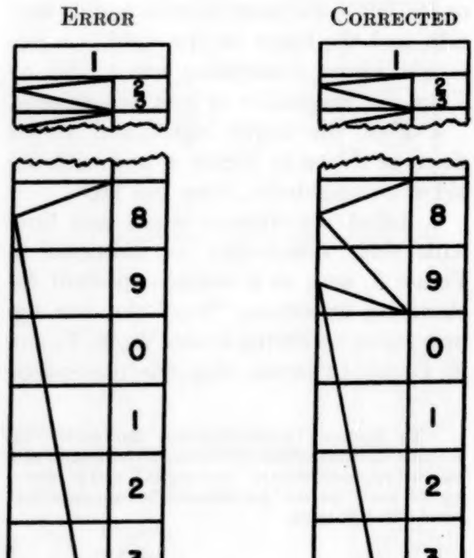
I am an instructor at the General Instructor School at Sampson Air Force Base. It is the mission of the school to prepare enlisted men for their careers as U.S. Air Force instructors. My particular responsibility is that of preparing and training those men who will be the mathematics instructors of basic trainees.

In the November 1954 issue of THE MATHEMATICS TEACHER (vol. XLVII, pages 482–487) there is an article which you prepared. I became very interested as I read it, and spent several hours discovering the principles of the tables described therein. It may interest you to know how I came to have a solid knowledge of the Napier and the Genaille tables. I first realized that this was all probably connected to a common multiplication table, and I drew one up. Then I realized that if I cut along the vertical columns, I could rearrange them to have their headings form the number I wished to multiply. A closer look at the Napier procedure revealed that this was exactly the procedure. I then made a set of tables using the diagonal line as did Napier. On the first computation with them, I discovered how much of a time saver it was to be able to move across the strips diagonally in carrying.

Feeling myself worked into a corner, I turned to the Genaille rods. After several minutes of very close scrutiny, I discovered the principle of the angular lines, and proceeded to construct a Genaille-type table as Professor Larsen had done. After I had made each horizontal grouping, I checked to see if Larsen's drawings were the same as mine, and, as I worked through the tables, I discovered the error which you hinted might be there. It is in the column marked "1" and opposite the index "8" on the left side of the chart. The enclosed drawing shows the correction.

I must say that I thoroughly enjoyed your article, and commend you for writing it in such a way as to stimulate others to inquire as to the construction and principles thereof. I am anxious to see emphasized the process of learning of mathematical concepts by "discovery" because I feel the generalizations thus obtained are more lasting and yet more open to insightful extension.

Sincerely yours,
Robert R. Wright
A/2c USAF



• MATHEMATICAL MISCELLANEA

Edited by Paul C. Clifford, State Teachers College, Montclair, New Jersey, and Adrian Struyk, Clifton High School, Clifton, New Jersey

A table of integral solutions of $a^2 + b^2 + c^2 = r^2$

by Francis L. Miksa, Aurora, Illinois

Although there are several standard identities which lead to parametric solutions[1,3,5] of the diophantine equation

$$(1) \qquad a^2+b^2+c^2=r^2,$$

the writer used, instead of one of these, a systematic method designed to yield all sets a, b, c corresponding to a given r. The solutions tabulated here were obtained by using a Friden calculating machine, a partition table listing solutions of $a^2+b^2=n$, and a factor table. The method was to subtract successive squares $1^2, 2^2, 3^2, \cdots, k^2$ from r^2, by entering r^2 in the machine and then subtracting successively the odd integers $1, 3, 5, \cdots, 2k-1$. This is effective because the sum of the first k odd integers is equal to k^2. After each subtraction the remainder was, therefore, a difference of two squares, r^2-c^2. If this could be expressed as a sum of two squares, a^2+b^2, then a solution of (1) was at hand. By means of the factor table, r^2-c^2 was analyzed for its factor composition. If there appeared any factor of the form $4m-1$ to an odd power, then r^2-c^2 could not be expressed as a sum of two squares. If, however, the factors were of the form $4m+1$, and/or $4m-1$ form factors entered as even powers, then r^2-c^2 was expressible as a sum of two squares in one or more ways.[4,6] In this case the partition table was used to find such representations, yielding one or more solutions of (1).

Comparison of the present table with the earlier table[2] of R. H. Bacon (extending to $r=99$) reveals fourteen errors in Bacon's results. Corrections to be made are tabulated below, and are indicated by italics.

	a	*b*	*c*	*r*
change	24	*29*	48	61
change	26	38	*43*	63
change	*3*	12	76	77
change	24	*27*	68	77
change	*16*	47	64	81
delete	32	51	52	81
insert	32	51	60	85
change	6	*54*	73	91
insert	9	62	66	91
delete	18	51	58	93
change	16	*63*	72	97
change	24	*33*	88	97
delete	32	51	72	97
insert	17	46	86	99

With these corrections Bacon's table contains 347 solutions, the required number of primitive solutions for $r \leqq 100$.[7]

[1] Bacon, R. H., "Integral Solutions of $x^2+y^2+z^2=r^2$," *School Science and Mathematics*, vol. 47 (Feb. 1947), pp. 155–159.

[2] *Ibid.*, pp. 159–162.

[3] Carmichael, R. D., *Diophantine Analysis* (New York: Wiley & Sons, 1915), p. 38.

[4] *Ibid.*, pp. 39–43.

[5] Dickson, L. E., *History of the Theory of Numbers*, Vol. II (Washington: Carnegie Institute, 1923; Reprint, New York: Chelsea, 1952), chap. 7, "Sum of Three Squares."

[6] *Ibid.*, chap. 6, "Sum of Two Squares."

[7] Lehmer, D. H., *Guide to Tables in the Theory of Numbers* (Washington: National Research Council, 1941), p. 62.

Table of integral solutions of $a^2+b^2+c^2=r^2$ for all odd values of r from $r=3$ to $r=207$.
Asterisks indicate non-primitive sets.

r	a	b	c
r=3	1	2	2
r=5	none		
r=7	2	3	6
r=9	1	4	8
	3	6	6*
	4	4	7
r=11	2	6	9
	6	6	7
r=13	3	4	12
r=15	2	5	14
	2	10	11
	5	10	10*
r=17	1	12	12
	8	9	12
r=19	1	6	18
	6	6	17
	6	10	15
r=21	4	5	20
	4	8	19
	4	13	16
	6	9	18*
	7	14	14*
	8	11	16
r=23	3	6	22
	3	14	18
	6	13	18
r=25	9	12	20
	12	15	16
r=27	2	7	26
	2	10	25
	2	14	23
	3	12	24*
	7	14	22
	9	18	18*
	10	10	23
	12	12	21*
r=29	3	16	24
	11	12	24
	12	16	21
r=31	5	6	30
	6	14	27
	6	21	22
	14	18	21
r=33	1	8	32
	4	7	32
	4	17	28
	6	18	27*
	7	16	28
	8	8	31
	8	20	25
	11	22	22*
	17	20	20
	18	18	21*
r=35	1	18	30
	6	10	33
	6	17	30
	10	15	30*
	15	18	26
r=37	3	8	36
	3	24	28
	8	24	27
	12	21	28
r=39	2	19	34
	2	26	29
	9	12	36*
	10	14	35
	13	14	34
	13	26	26*
	14	22	29
	19	22	26
r=41	4	12	39
	4	24	33
	9	24	32
	12	24	31
	23	24	24
r=43	2	9	42
	2	18	39
	6	7	42
	7	30	30
	9	18	38
	18	25	30
r=45	4	28	35
	5	8	44
	5	20	40*
	6	15	42*
	6	30	33*
	8	19	40
	13	16	40
	15	30	30*
	16	20	37
	20	20	35*
	20	28	29
r=47	2	21	42
	6	18	43
	6	27	38
	11	18	42
	18	21	38
	18	27	34
r=49	4	9	48
	4	33	36
	9	32	36
	12	24	41
	12	31	36
	14	21	42*
	15	24	40
	23	24	36
r=51	1	10	50
	1	22	46
	1	34	38
	2	14	49
	3	36	36*
	10	10	49
	14	14	47
	14	17	46
	14	31	38
	17	34	34*
	22	31	34
	24	27	36*
r=53	8	12	51
	8	21	48
	12	19	48
	12	27	44
	12	36	37
	27	28	36
r=55	3	10	54
	3	30	46
	6	35	42
	10	18	51
	10	30	45*
	18	26	45
	19	30	42
	30	30	35*
r=57	3	18	54*
	4	23	52
	4	32	47
	7	8	56
	7	40	40
	8	28	49
	16	17	52
	16	28	47
	17	32	44
	18	18	51*
	18	30	45*
	19	38	38*
	23	28	44
	25	32	40
	28	28	41
r=59	6	9	58
	6	14	57
	6	23	54
	6	41	42
	9	22	54
	9	30	50
	14	39	42
	30	30	41
r=61	3	24	56
	11	36	48
	12	21	56
	20	36	45
	21	24	52
	24	29	48
	24	36	43
r=63	2	11	62
	2	22	59
	2	34	53
	2	43	46
	5	10	62
	5	38	50
	7	28	56*
	10	37	50
	11	22	58
	12	15	60*
	12	24	57*
	12	39	48*
	18	27	54*
	21	42	42*
	22	26	53
	22	37	46
	24	33	48*
	26	38	43
	28	28	49*
	34	37	38
r=65	7	24	60
	15	20	60*
	15	36	52
	20	24	57
	20	39	48
	25	36	48
r=67	6	22	63
	6	33	58
	14	18	63
	15	30	58
	15	42	50
	18	42	49
	22	33	54
	30	33	50
	31	42	42
r=69	4	11	68
	4	16	67
	4	32	61
	4	44	53
	5	40	56
	9	18	66*
	9	42	54*
	11	44	52
	16	28	61
	16	32	59
	16	37	56
	18	39	54*
	20	35	56
	23	46	46*
	28	29	56
	35	40	44
r=71	3	26	66
	3	46	54
	10	45	54
	18	19	66
	18	46	51
	19	42	54
	26	42	51
	30	35	54
	30	45	46
r=73	1	12	72
	8	9	72
	8	36	63
	12	12	71
	12	33	64
	12	44	57
	24	28	63
	28	36	57
	33	44	48
r=75	7	10	74
	7	26	70
	10	14	73
	10	22	71
	10	25	70*
	10	41	62
	10	50	55*
	14	23	70
	22	46	55
	25	34	62
	25	50	50*
	27	36	60*
	34	38	55
	36	45	48*
	38	41	50
r=77	3	12	76
	3	36	68
	4	27	72
	5	48	60
	12	32	69
	12	36	67
	12	48	59
	13	24	72
	14	42	63*
	22	33	66*
	24	27	68
	27	40	60
	32	48	51
	40	45	48
	42	42	49*
r=79	6	11	78
	6	27	74
	6	38	69
	6	43	66
	11	42	66
	18	21	74
	18	34	69
	21	30	70
	21	38	66
	27	34	66
r=81	1	28	76
	1	44	68
	6	21	78*
	6	30	75*
	6	42	69*
	8	16	79
	8	49	64
	9	36	72*
	16	23	76
	16	41	68
	16	47	64
	17	56	56
	20	44	65
	20	55	56
	21	42	66*
	23	44	64
	27	54	54*
	28	41	64
	30	30	69*
	32	49	56
	36	36	63*
	40	44	55
r=83	2	18	81
	2	54	63
	15	42	70
	18	18	79
	18	33	74
	18	47	66
	30	33	70
	30	42	65
	33	38	66
	34	42	63
	42	47	54
r=85	4	45	72
	5	12	84
	5	60	60*
	12	59	60
	21	40	72
	24	32	75
	24	45	68
	32	51	60
	40	45	60*
r=87	2	13	86
	2	26	83
	2	29	82
	2	61	62
	9	48	72*
	13	26	82
	13	50	70
	14	22	83
	14	38	77
	19	22	82
	19	58	62
	22	34	77
	22	58	61
	29	58	58*
	33	36	72*
	35	38	70
	35	50	62
	36	48	63*
r=89	8	36	81
	9	28	84
	15	36	80
	15	60	64
	17	24	84
	24	28	81
	24	48	71
	24	57	64
	36	36	73
	36	55	60
	39	48	64
r=91	6	18	89
	6	26	87
	6	39	82
	6	54	73
	9	10	90
	9	46	78
	9	62	66
	18	54	71
	21	28	84*
	26	39	78*
	26	57	66
	30	55	66
	39	54	62
	46	54	57
r=93	4	13	92
	4	52	77
	8	11	92
	8	29	88
	8	53	76
	8	64	67
	11	28	88
	13	52	76
	15	18	90*
	18	42	81*
	18	63	66*
	20	32	85
	20	43	80
	28	44	77
	31	62	62*
	32	35	80
	32	43	76
	32	56	67
	42	54	63*
	43	52	64
	52	53	56
r=95	5	30	90*
	5	54	78
	6	42	85
	6	58	75
	14	27	90
	21	22	90
	21	50	78
	22	54	75
	27	30	86
	30	30	85*
	30	50	75*
	30	58	69
	42	50	69
r=97	7	12	96
	7	48	84
	12	39	88
	12	47	84
	12	52	81
	16	63	72
	24	33	88
	25	60	72
	33	56	72
	39	52	72
	47	60	60
	48	56	63
r=99	1	14	98
	1	70	70
	2	31	94
	2	49	86
	3	24	96*
	10	26	95
	10	65	74
	11	44	88*
	12	21	96*
	12	51	84*
	14	14	97
	14	47	86
	14	58	79
	17	26	94
	17	46	86
	18	54	81*
	21	48	84*
	24	24	93*
	24	60	75*
	26	49	82
	26	65	70
	31	38	86
	31	46	82
	31	58	74
	33	66	66*
	38	46	79
	44	44	77*
	46	47	74
	49	50	70
	51	60	60*
	54	54	63*
r=101	3	56	84
	12	16	99
	12	29	96
	16	27	96
	16	36	93
	16	69	72
	21	36	92
	21	52	84
	29	48	84
	36	43	84
	36	61	72
	48	56	69
r=103	3	14	102
	3	42	94
	3	50	90
	6	13	102
	6	67	78
	13	66	78
	14	42	93
	18	22	99
	18	66	77
	22	45	90
	42	54	77
	42	66	67
	45	50	78
r=105	1	32	100
	1	68	80
	3	54	90*
	4	20	103
	4	40	97
	8	31	100
	8	44	95
	14	35	98*
	14	70	77*
	16	55	88
	18	30	99*
	18	51	90*
	20	25	100*
	20	40	95*
	20	52	89
	20	65	80*
	23	64	80
	25	44	92
	25	68	76
	30	45	90*
	31	40	92
	32	65	76
	35	70	70*
	40	41	88
	40	55	80*
	40	64	73
	45	54	78*
	52	64	65
r=107	6	62	87
	6	73	78
	9	42	98
	10	18	105
	10	57	90
	18	39	98
	18	55	90
	18	71	78
	39	62	78
	42	42	89
	42	46	87
	42	66	73
	46	57	78
	57	62	66
r=109	8	51	96
	8	69	84
	12	44	99
	19	48	96
	27	44	96
	27	64	84
	28	36	99
	35	60	84
	36	37	96
	36	44	93
	36	48	91
	44	69	72
	51	64	72
r=111	2	46	101
	2	59	94
	5	14	110
	5	70	86
	9	24	108*
	9	72	84*
	10	11	110
	11	58	94

11	74	82
14	26	107
14	62	91
14	70	85
24	72	81*
26	38	101
26	53	94
26	58	91
36	63	84*
37	46	94
37	74	74*
38	59	86
46	59	82
53	46	86
58	59	74

$r=113$

4	33	108
4	72	87
9	12	112
9	32	108
12	31	108
12	40	105
12	60	95
23	24	108
23	72	84
24	68	87
32	72	81
33	68	84
40	60	87
49	72	72

$r=115$

2	15	114
2	30	111
6	33	110
15	30	110*
15	42	106
15	70	90*
30	33	106
30	54	97
30	65	90*
30	78	79
34	63	90
42	70	81
47	54	90
54	65	78
63	66	70

$r=117$

4	77	88
6	57	102*
6	78	87*
8	13	116
8	20	115
8	53	104
8	80	85
11	32	112
13	52	104*
13	76	88
19	28	112
20	80	83
27	36	108*
28	32	109
28	52	101
28	68	91
30	42	105*
32	43	104
32	76	83
39	42	102*
39	78	78*
42	66	87*
43	64	88
52	52	91*
52	64	83
53	56	88
56	68	77
57	66	78*

$r=119$

2	66	99
3	34	114
3	54	106
6	30	115
6	45	110
6	61	102
6	74	93
7	84	84*
18	29	114
18	51	106
29	54	102
30	45	106
34	51	102*
34	66	93
38	54	99
51	74	78
54	66	83
56	63	84*

61	66	78

$r=121$

4	15	120
4	48	111
4	60	105
4	84	87
15	60	104
22	66	99*
23	84	84
24	36	113
24	39	112
24	49	108
24	57	104
32	36	111
32	81	84
36	41	108
36	76	87
39	48	104
48	76	81
49	72	84
66	66	77*

$r=123$

2	22	121
2	55	110
7	14	122
7	34	118
7	62	106
7	74	98
12	36	117*
12	72	99*
14	73	98
22	22	119
22	71	98
22	82	89
23	26	118
23	50	110
23	58	106
25	70	98
26	62	103
27	72	96*
34	58	103
36	72	93*
41	62	98
41	82	82*
50	55	98
58	62	89
58	71	82
69	72	72*
70	70	73

$r=125$

3	80	96
12	45	116
12	60	109
19	60	108
21	28	120
21	72	100
24	60	107
24	80	93
28	75	96
35	72	96
44	45	108
45	60	100*
53	60	96
60	75	80*

$r=127$

10	30	123
10	75	102
18	26	123
18	46	117
18	53	114
18	66	107
21	42	118
21	78	98
26	42	117
30	75	98
37	42	114
37	66	102
42	54	107
42	69	98
42	78	91
53	54	102

$r=129$

1	16	128
1	64	112
6	27	126*
6	54	117*
7	56	116
7	76	104
8	44	121
16	16	127
16	76	103
16	89	92
18	21	126*
20	40	121
20	79	100
21	90	90*
27	54	114*
28	49	116
28	71	104
31	56	112
40	65	104
40	71	100
56	64	97
43	86	86*
44	44	113
44	64	103
44	79	92
49	76	92
54	75	90*
56	64	97
56	71	92
64	68	89
68	76	79

$r=131$

6	15	130
6	22	129
6	66	113
6	90	95
14	18	129
14	33	126
14	63	114
14	81	102
15	90	94
18	31	126
22	66	111
31	90	90
42	49	114
42	81	94
49	66	102
50	81	90
63	66	94

$r=133$

3	16	132
3	36	128
3	48	124
3	92	96
7	42	126*
11	12	132
12	88	99
16	48	123
20	60	117
20	75	108
24	27	128
24	92	93
27	88	96
29	72	108
36	52	117
36	88	93
38	57	114*
42	42	119*
42	70	105*
48	61	108
48	72	101
52	72	99
60	75	92
72	69	88

$r=135$

10	13	134
10	35	130*
10	50	125*
10	70	115*
10	83	106
12	84	105*
13	34	130
14	77	110
15	24	132*
15	60	120*
18	45	126*
18	90	99*
22	29	130
22	46	125
24	57	120*
29	50	122
34	38	125
34	62	115
35	46	122
35	70	110*
35	86	98
38	70	109
39	48	120*
45	90	90*
48	60	111*
50	50	115*
50	62	109
50	67	106
50	83	94
60	60	105*
60	84	87*
61	70	98
67	70	94
70	77	86

$r=137$

7	36	132
7	84	108
9	48	128
12	20	135
12	57	124
12	65	120
12	96	97
16	33	132
20	63	120
33	36	128
33	48	124
33	92	96
48	48	119
48	84	97
56	63	108
57	84	92
63	84	88

$r=139$

9	14	138
9	58	126
9	66	122
9	94	102
14	30	135
14	57	126
14	90	105
18	66	121
23	54	126
30	39	130
30	86	105
39	54	122
39	86	102
41	42	126
42	54	121
54	87	94
66	66	103
66	86	87

$r=141$

4	37	136
4	52	131
4	59	128
4	67	124
5	16	140
5	80	116
6	63	126*
16	44	133
16	80	115
16	88	109
18	54	129*
18	81	114*
19	32	136
19	56	128
28	44	131
28	61	124
28	91	104
32	59	124
33	54	126*
37	56	124
40	80	109
40	91	100
44	67	116
44	88	101
47	94	94*
52	61	116
54	63	114*
54	81	102*
56	77	104
56	91	92
59	80	100

$r=143$

6	37	138
6	82	117
21	78	118
21	98	102
26	27	138
26	78	117*
27	42	134
27	62	126
27	82	114
30	75	118
30	90	107
33	44	132*
37	78	114
42	53	126
42	69	118
42	91	102
54	62	117
54	78	107
69	78	98
75	82	90
78	78	91*

$r=145$

1	60	132
8	15	144
8	81	120
15	48	136
15	80	120*
24	55	132
28	60	129
28	96	105
36	60	127
36	73	120
48	80	111
55	60	120*
60	63	116
60	80	105*
63	84	100
80	84	87

$r=147$

1	38	142
1	82	122
2	17	146
2	34	143
2	74	127
2	94	113
10	22	145
10	97	110
12	27	144*
12	99	108*
17	34	142
17	58	134
17	86	118
22	26	143
22	31	142
22	65	130
22	79	122
22	95	110
27	96	108*
28	35	140*
28	56	133*
28	91	112*
31	82	118
36	72	123*
36	93	108*
38	47	134
38	79	118
38	86	113
42	63	126*
45	72	120*
46	58	127
46	82	113
47	50	130
47	74	118
49	98	98*
50	65	122
56	77	112*
58	74	113
58	94	97
69	72	108*
74	82	97

$r=149$

4	24	147
4	48	141
4	69	132
4	84	123
13	36	144
21	32	144
21	96	112
24	45	140
24	48	139
24	85	120
32	36	141
36	59	132
36	76	123
45	76	120
51	84	112
59	84	108
69	76	108
77	84	96

$r=151$

2	54	141
2	99	114
6	34	147
6	51	142
6	61	138
6	83	126
29	54	138
29	78	126
30	99	110
34	42	141
34	51	138
34	93	114
42	99	106
45	70	126
51	90	110
54	83	114
54	93	106
61	78	114
70	90	99

$r=153$

3	30	150*
3	66	138*
3	102	114*
4	17	152
4	68	137
6	42	147*
7	16	152
7	104	112
9	108	108*
16	52	143
16	103	112
17	68	136*
17	88	124
25	80	128
28	40	145
28	55	140
28	79	128
28	92	119
30	30	147*
32	44	143
32	88	121
40	47	140
42	42	141*
42	51	138*
42	93	114*
44	52	137
47	52	136
47	88	116
49	68	128
49	92	112
51	102	102*
52	73	124
66	93	102*
68	68	119*
68	73	116
68	79	112
72	81	108*
80	80	103
88	88	89

$r=155$

2	39	150
2	105	114
7	90	126
9	38	150
9	70	138
18	30	151
18	74	135
25	66	138
25	102	114
30	25	150*
30	66	137
30	70	135*
30	105	110*
38	66	135
39	102	110
42	90	119
42	105	106
57	70	126
66	87	110
70	90	105*

$r=157$

12	13	156
12	48	149
12	51	148
12	68	141
12	72	139
12	104	117
13	84	132
21	48	148
21	108	112
36	77	132
40	75	132
40	93	120
48	72	131
48	99	112
51	68	132
68	75	120
72	76	117
72	93	104
77	84	108

$r=159$

10	34	155
10	85	134
10	91	130
11	14	158
11	38	154
11	62	146
11	106	118
14	37	154
14	86	133
14	101	122
22	59	146
22	74	139
24	36	153*
24	63	144*
34	53	146
34	85	130
34	101	118
36	57	144*
36	81	132*
36	108	111*
43	46	146
43	74	134
46	62	139
46	74	133
46	91	122
53	106	106*
59	62	134
59	70	130
62	101	106
70	91	110
74	98	101
81	84	108*
86	91	98

$r=161$

1	72	144
8	24	159
8	39	156
8	96	129
12	71	144
12	111	116
17	36	156
21	42	154*
21	98	126*
24	44	153
24	89	132
24	96	127
33	64	144
36	39	152
36	60	145
36	80	135
39	100	120
42	91	126*
44	57	144
44	81	132
44	108	111
46	69	138*
57	96	116
60	64	135
60	89	120
60	100	111
64	72	129
71	96	108

$r=163$

1	18	162
6	17	162
6	78	143
6	102	127
10	15	162
10	63	150
15	62	150
17	102	126
18	18	161
18	98	129
18	111	118
33	42	154
33	98	126
42	66	143
62	78	129
62	102	111
63	82	126
63	98	114
66	78	127
78	81	118
81	98	102

$r=165$

5	40	160*
5	64	152
5	104	128
8	56	155
8	100	131
9	30	162*
9	90	138*
16	20	163
16	37	160
18	105	126*
20	35	160*
20	61	152
20	68	149
20	85	140*
20	107	124
22	55	154*
22	110	121*
28	29	160
28	104	125

29	100	128
30	54	153*
30	90	135*
35	64	148
35	80	140*
35	112	116
40	40	155*
40	61	148
40	92	131
40	100	125*
43	76	140
43	100	124
54	78	135*
55	110	110*
56	67	140
56	80	133
56	92	125
57	90	126*
68	85	124
76	100	107
80	91	112
85	100	100*
90	90	105*

$r=167$

3	18	166
3	54	158
3	94	138
3	114	122
14	78	147
14	93	138
18	51	158
18	54	157
18	102	131
30	45	158
30	67	150
45	58	150
51	102	122
54	58	147
54	77	138
58	93	126
66	67	138
66	93	122
67	102	114
77	78	126
93	94	102

$r=169$

9	16	168
9	88	144
12	39	164
12	56	159
16	63	156
16	87	144
16	108	129
24	33	164
24	52	159
24	72	151
24	96	137
25	60	156
33	56	156
39	52	156*
39	108	124
52	96	129
56	105	120
60	65	144
63	96	124
72	96	119

$r=171$

2	26	169
2	41	166
2	89	146
7	74	154
7	106	134
9	54	162*
12	69	156*
12	96	141*
14	22	169
14	34	167
14	73	154
14	119	122
19	76	152*
21	24	168*
21	120	120*
22	71	154
23	34	166
23	86	146
24	84	147*
25	70	154
26	26	167
26	103	134
26	118	121
34	89	142
34	118	119
41	62	154
41	86	142
41	98	134
48	51	156*
48	84	141*
50	55	154
50	110	121
51	96	132*
54	54	153*
54	90	135*
55	70	146
57	114	114*
58	89	134
58	106	121
62	106	119
69	84	132*
71	110	110
74	98	119
75	96	120*
76	76	133*
84	84	123*
86	89	118
86	103	106

$r=173$

20	48	165
20	120	123
24	88	147
24	108	133
27	48	164
27	60	160
27	92	144
29	108	132
45	48	160
48	60	155
48	83	144
48	88	141
48	101	132
48	115	120
52	99	132
56	72	147
56	108	123
69	88	132
72	84	133
84	88	123
92	99	108

$r=175$

5	18	174
5	66	162
5	90	150*
6	58	165
6	75	158
6	117	130
18	90	149
22	30	171
22	54	165
27	86	150
27	114	130
30	50	165*
30	59	162
30	69	158
30	85	150*
30	102	139
50	75	150*
50	114	123
58	69	150
59	90	138
63	84	140*
66	85	138
75	90	130*
84	105	112*
85	102	114
86	90	123
90	94	117

$r=177$

4	97	148
4	112	137
8	17	176
8	41	172
8	52	169
8	92	151
8	104	143
8	113	136
17	112	136
18	27	174*
18	42	171*
18	69	162*
18	123	126*
20	73	160
20	95	148
27	66	162*
27	90	150*
28	31	172
28	68	161
28	88	151
28	119	128
31	92	148
41	88	148
42	117	126*
49	112	128
52	52	161
52	55	160
52	95	140
55	80	148
56	88	143
59	118	118*
64	73	148
64	92	137
68	112	119
73	80	140
73	112	116
79	112	112
88	104	113
90	90	123*
92	97	116

$r=179$

1	42	174
1	114	138
6	26	177
6	78	161
6	82	159
6	126	127
15	54	170
15	90	154
17	126	126
26	33	174
26	78	159
26	111	138
33	114	134
42	66	161
42	81	154
42	111	134
54	62	159
54	90	145
54	114	127
52	111	126
63	66	154
81	98	126
82	111	114

$r=181$

12	24	179
12	91	156
12	109	144
19	108	144
21	64	168
21	96	152
24	51	172
24	107	144
24	117	136
37	84	156
44	51	168
44	60	165
44	96	147
45	80	156
45	100	144
51	84	152
51	108	136
53	96	144
64	84	147
75	80	144
84	91	132
96	108	109

$r=183$

2	19	182
2	38	179
2	77	166
2	94	157
9	72	168*
10	67	170
10	115	142
13	14	182
13	98	154
14	58	173
14	82	163
19	38	178
33	108	144*
35	58	170
35	110	142
36	63	168*
38	46	173
38	67	166
38	74	163
38	86	157
38	109	142
38	122	131
46	82	157
58	74	157
58	115	130
60	108	135*
61	98	142
61	122	122*
62	77	154
63	72	156*
67	94	143
67	110	130
72	87	144*
72	108	129*
77	86	142
77	98	134
82	98	131
82	109	122

$r=185$

12	15	184
12	41	180
15	40	180*
15	76	168
15	120	140*
23	36	180
36	40	177
36	48	175
36	105	148
40	96	153
40	120	135*
48	111	140
49	60	168
49	120	132
60	105	140*
72	104	135
72	120	121
76	105	132
84	87	140
84	113	120
96	103	120

$r=187$

6	38	183
6	57	178
7	18	186
7	126	138
9	42	182
11	132	132*
18	34	183
18	39	182
18	78	169
18	106	153
18	126	137
25	90	162
34	87	162
34	102	153*
38	57	174
38	105	150
39	42	178
42	102	151
42	119	138
57	74	162
57	102	146
70	87	150
70	105	138
74	78	153
78	78	151
78	87	146
88	99	132*
90	90	137
102	102	119*

$r=189$

4	19	188
4	43	184
4	76	173
4	128	139
6	33	186*
6	66	177*
6	102	159*
6	129	138*
8	116	149
8	131	136
11	16	188
11	68	176
11	100	160
14	49	182*
14	70	175*
14	98	161*
15	30	186*
15	114	150*
16	52	181
16	67	176
16	104	157
19	76	172
19	92	164
19	116	148
21	84	168*
28	32	184
29	128	136
30	111	150*
32	44	181
32	61	176
32	124	130
33	66	174*
36	45	180*
36	72	171*
36	117	144*
40	85	164
40	125	136
43	124	136
44	53	176
44	83	164
44	109	148
49	98	154*
52	104	149
54	81	162*
61	80	160
63	126	126*
66	78	159*
66	111	138*
70	70	161*
72	99	144*
76	88	149
76	107	136
78	114	129*
80	100	139
83	116	124
84	84	147*
85	100	136
88	104	131
92	109	124
102	111	114*
104	107	116

$r=191$

6	43	186
6	69	178
6	101	162
6	123	146
11	42	186
11	78	174
21	38	186
21	54	182
21	66	178
21	126	142
27	34	186
27	74	174
34	78	171
34	90	165
38	69	174
42	74	171
43	66	174
54	126	133
66	70	165
66	90	155
69	74	162
69	102	146
74	123	126
101	102	126

$r=193$

12	57	184
12	76	177
12	96	167
12	113	156
15	68	180
15	132	140
33	48	184
33	72	176
36	68	177
36	112	153
40	57	180
40	132	135
48	63	176
48	84	167
48	103	156
49	132	132
57	76	168
57	120	140
63	112	144
68	96	153
68	120	135
72	121	132
84	113	132
96	103	132

$r=195$

1	70	182
2	86	175
2	110	161
10	17	194
10	26	193
10	38	191
10	79	178
10	95	170*
10	130	145*
17	94	170
21	72	180*
26	65	182*
26	82	175
26	130	143*
31	110	158
31	130	142
38	95	166
45	60	180*
45	108	156*
46	65	178
46	122	145
47	110	154
49	50	182
49	82	170
50	70	175*
50	98	161
60	72	171*
60	117	144*
65	70	170*
65	94	158
65	130	130*
70	97	154
70	110	145*
74	110	143
75	108	144*
79	122	130
86	98	145
94	95	142
95	110	130*

$r=197$

3	44	192
3	80	180
3	96	172
24	37	192
24	93	172
24	108	163
24	123	152
28	48	189
28	75	180
28	99	168
28	117	156
35	72	180
37	96	168
44	93	168
48	91	168
53	60	180
53	108	156
60	75	172
72	80	165
72	84	163
72	123	136
84	93	152
93	108	136
108	116	117

$r=199$

3	126	154
6	19	198
6	62	189
6	114	163
6	134	147
19	114	162
26	30	195
26	93	174
26	138	141
30	51	190
30	99	170
35	90	174
35	126	150
42	46	189
42	134	141
46	126	147
51	82	174
51	90	170
51	134	138
62	126	141
78	99	154
82	114	141
90	125	126
93	114	134
99	118	126

$r=201$

1	20	200
1	104	172
1	136	148
4	28	199
4	56	193
4	71	188
4	97	176
7	44	196
7	116	164
18	66	189*
18	99	174*
20	20	199
20	76	185
20	95	176
28	56	191
31	32	196
31	64	188
31	92	176
31	112	164
32	104	169
32	116	161
41	88	176
42	54	189*
44	112	161
45	90	174*
45	126	150*
54	126	147*
55	80	176
56	119	152
56	136	137
64	73	176
64	88	169
64	97	164
66	99	162*
67	134	134*
71	76	172
71	92	164
71	116	148
76	95	160
76	127	136
88	119	136
90	99	150*
93	126	126*
104	104	137
104	116	127

$r=203$

9	18	202
9	122	162
18	41	198
18	57	194
18	86	183
18	103	174
18	114	167
18	121	162
21	112	168*
22	18	201
22	39	198
22	135	150
23	78	186
23	102	174
30	78	185
30	130	153
54	102	167
54	122	153
57	58	186
57	102	166
57	114	158
58	66	183
58	87	174*
66	103	162
77	84	168*
78	87	166
78	130	135
84	112	147*
86	87	162
86	102	153
90	103	150
90	122	135

$r=205$

3	20	204
3	60	196
20	60	195*
20	69	192
20	120	165*
24	115	168
24	132	155
27	36	200
27	120	164
36	52	195
36	123	160
45	56	192
45	120	160*
52	60	189
56	108	165
60	101	168
60	120	155*
60	133	144
69	108	160
83	120	144
84	88	165
88	120	141
101	120	132
115	120	120*

$r=207$

5	82	190
5	118	170
11	98	182
12	33	204*
12	48	201*
12	96	183*
12	132	159*
14	38	203
14	43	202
14	62	197
14	133	158*
15	120	168
22	34	203
22	86	187

22	98	181	37	118	166	50	85	182	82	85	170
22	142	149	37	142	146	50	107	170	82	110	155
23	92	184*	38	91	182	54	117	162*	82	118	149
26	37	202	38	133	154	58	82	181	84	87	168*
26	58	197	43	58	194	58	139	142	91	98	158
27	54	198*	43	70	190	60	105	168*	92	92	161*
27	126	162*	43	110	170	62	107	166	98	118	139
33	132	156*	48	84	183*	62	133	146	105	120	132*
34	82	187	48	96	177*	69	138	138*	106	107	142
37	62	194	48	111	168*	70	118	155	106	118	133

Present-day activity in numerical analysis[1]

by Fred Gruenberger, Numerical Analysis Laboratory, University of Wisconsin, Madison, Wisconsin

In the past, it has almost invariably been true that mathematical research has preceded its need by a long period, frequently more than a century. As a small example, number theorists had developed arithmetic in systems other than decimal (e.g., binary) many years ago. This "useless" bit of mathematics was ready and waiting when binary automatic calculators came into being.

The advent of high-speed calculators (since 1945 or so) has pointed up large uncharted areas of mathematics which need intensive research.

Consider, for example, the solution of systems of linear algebraic equations. For small sized systems we might conceivably solve by determinants, or by elimination, as is taught in high-school algebra classes. For systems of order greater than four, more rapid and convenient methods are needed, and a great many such methods are known. Frequently the choice of method depends on the particular machine to be used. Problems come from all directions: how much intermediate storage will be needed? how many instructions are needed per root? what accuracy can we expect in the roots? what will happen if the system happens to have a zero in a particular position? There is no one answer. Different methods may be used on different systems.

[1] See Paul Brock, "Mathematicians and Automata," THE MATHEMATICS TEACHER, XLVII (December, 1954), p. 514.

Only a few years ago it was considered good for an automatic calculator to be able to solve a system of eight equations in eight unknowns. Today many laboratories consider it a routine problem to solve systems of order 40 to 60. And the need has already arisen to solve systems of order 2000.

In this problem, and almost all other numerical problems, rounding errors cause trouble whose nature is not, at present, well understood. A calculating machine generally treats all numbers as integers; irrational and transcendental numbers must obviously be truncated, and an error is immediately introduced into the calculation. But even repeating decimals must be curtailed. At nearly every stage of a long calculation a rounding error occurs. There is a tacit faith in numerical work that rounding errors will tend to cancel out. We have no assurance that they will, however.

Much research, then, is called for on numerical methods. At the same time, research is needed into the operation and construction of the machines themselves. At the present stage of the computing art, machines exist which can execute a million operations (fixed decimal) per minute. This can and will be increased eventually; however, at present there is a severe limitation on input and output devices and on means of obtaining high-speed storage. In simple terms, it is of no use to calculate

at great speed unless intermediate results can be stored and recalled at the same speed. And again, given the speed we need, the problem remains of getting a large amount of such storage. Today the largest portion of research time on machines is probably spent on these problems.

Assuming that a machine is in production, problems still remain as to how best to operate it. Most machines so far constructed are essentially fixed decimal. For scientific purposes, however, it is desirable to operate them with scientific notation (e.g., a mile is 6.336×10^4 inches), or "floating decimal" as it is called. Yet people tend to think in terms of no particular notation —the problem is how best to make the machine float. In floating decimal work, arithmetic is not commutative; i.e., AB is not equal, always, to BA. (The difference is then in the last digit.)

The problems are virtually endless, and very little has been done on them. The over-all picture, however, is quite bright. There are today more than 60 giant calculators in operation in this country (nearly all of them 24 hours a day) and over 5000 small ones. Great strides have been taken in the art, and new developments are appearing every day.

Changes that can be expected in the near future are these:

1. The principle of "stored programming" will extend to the small installation. This is simply the ability, on a machine, to store the machine's instructions in the same way and in the same place as the numbers of the problem. At present, this ability (which is a powerful tool in numerical work) is limited to the giant machines. Small sized stored program machines will become available in quantity.

2. Methods of analysis will undoubtedly be improved and various procedures will become standardized.

3. Problems of input, output, and storage will be solved to a better extent than at present.

College recruiting trends

Here are two interesting excerpts from the 1955 Midwest College Placement association's "College Recruiting Survey."

1. How will your 1955 requirements for college-level men compare with those for preceding years?

	1955 vs. 1954 *No. of Companies*		1954 vs. 1953 *No. of Companies*	
	Tech.	*Non-Tech.*	*Tech.*	*Non-Tech.*
More	32	17	32	15
The Same	102	101	69	68
Less	26	30	32	18
	160	148	133	101

2. As nearly as you can now estimate, what will your 1955 gross monthly salary offering be for men without previous business or industrial experience?

1955—Non-Veteran *Salary Range*	*No. of Companies* *Technical* *B.S.*	*M.S.*	*Ph.D.*	*Non-Technical* *Bach.*	*Master*
\$250 or under	1	—	—	5	1
251–275	1	—	—	3	—
276–300	7	—	—	24	4
301–325	13	2	—	44	11
326–350	43	12	3	41	22
351–375	67	19	1	22	18
376–400	26	34	2	4	20
401–425	—	39	3	—	8
426–450	1	9	7	—	—
451–475	—	—	1	—	—
476–500	—	—	7	—	—
501–525	—	—	11	—	—
526–550	—	—	21	—	—
551–575	—	—	12	—	—
576–600	—	—	4	—	—
601–625	—	—	1	—	—
	159	115	73	143	84

Taken from Engineering and Scientific Manpower Newsletter, *#70, December 15, 1954*

Edited by Lucien B. Kinney, Stanford University, Stanford, California, and Dan T. Dawson, Stanford University, Stanford, California

Who teaches algebra to whom?

by George Truscott, Ray Lyman Wilbur Junior High School, Desmond T. Jenkins, Ray Lyman Wilbur Junior High School, Palo Alto, California, and Connell Korb, Counselor, David Starr Jordan Junior High School, Palo Alto, California

The mathematics program in our public schools presents a whole series of paradoxes. With our culture becoming increasingly quantitative in all its aspects, our adult population continues to be mathematically illiterate. With our world leadership and the survival of a democratic civilization dependent on our technical supremacy, our supply of technical leaders is drying up at the source. The series of paradoxes extends down to specifics of the program: while our high school graduates are under criticism for arithmetic deficiencies, there is parental pressure for pupils to choose algebra and geometry rather than general mathematics.

The present paper grew out of the experiences of three members of a junior high school curriculum committee as they undertook to deal with these paradoxes. Having reached some conclusions as to what needs to be done, they desire to present their ideas for criticism and suggestion.

First, as to the problem as it manifests itself in the junior high school, we believe that this can best be presented as an allegory, with the understanding that any resemblance to persons living or dead is not only unintentional, but incredible.

A GUIDANCE ALLEGORY

Once upon a time there was a set of triplets, Andrew, Bertram and Catherine. The time came for them to go to junior high school. Because they were triplets they had always been together in elementary school. It seemed a shame to separate them now, so they were all assigned to the same seventh-grade math class; and in the math room, because their last name began with the same letter, they all sat together in the same row.

After discussing what they had done during vacation, they turned to the first page of the book and began the seventh-grade work. Now they were doing the advanced work their sixth-grade teacher had promised. Everyone reviewed addition, subtraction, multiplication and division. That night Bertram spent twenty minutes on his homework, and Catherine had finished hers in class, but Andrew, who was expected to do well in mathematics because he was a boy, spent two hours on his and then missed half.

Just before Christmas vacation, the teacher found the sixth-grade achievement tests. Catherine scored 9.2; Bertram scored 7.1; and poor Andrew who never could abstract, 3.2. This bothered the teacher, but she was on page 86 in the text and had to cover page 104 by Friday, so she had no time to go back. Andrew spent three hours on his homework on the occasions that his parents became threatening and even then missed three-fourths.

The teacher was disturbed. One day as she went down the hall she met the shop teacher who kept the counseling files. She told him about Andrew, and the shop teacher promised to do something at once; but because he was still recording the weights and heights on the personal record

cards, he didn't quite have time to get around to Andrew.

The days passed. A, B, and C spent eight weeks on percentage. More days passed. A, B, and C spent weeks on perimeters and areas. By Easter they had covered word problems and could do almost any of those where coffee cost 29 cents a pound, and eggs were 15 cents a dozen. They had all measured their bedrooms with a ruler. Catherine, who, contrary to all reasonable expectation, was good in math, remembered the dimensions from the last two years and did not spend any time measuring. Bertram was a whizz with the tape measure and finished in twenty minutes, but poor Andrew couldn't quite manage twelve inches and after suffering through the regular two hours of homework, and having moved his bed three times, was quite sure that his room was $7' \times 26'$.

When the spring achievement tests were scored, Catherine, who was a girl and not expected to do too well, scored 10.3, Bertram scored 7.9, and Andrew scored 3.1. This might have offered some problems, but Andrew was so meek and cooperative that he found himself sitting again in front of his brother and sister in the eighth-grade class.

The triplets' eighth-grade year was uneventful. After reviewing addition, multiplication, subtraction and division in chapter one of the eighth-grade text, they proceeded to apply what they had learned in the preceding grades.

By Christmas they had been introduced to equations and signed numbers and by Easter they were doing problems involving volume and complex figures. Bertram, following Catherine's example, had kept the dimensions of his bedroom, and had no homework, but Andrew was laboriously and sullenly trying to figure out how to fit his bed in a $4' \times 29'$ room.

The time had now come for the counselor to counsel. Shop work was well under way and the assignments for ninth-grade math had to be on the principal's desk not later than the next Monday. Bertram, who wanted to be a professional baseball player, got algebra. Catherine, who was much interested in math and economics was discouraged from such aspirations because as a girl she had no technical future. She was signed up for general mathematics. Andrew, who was interested in literature, poetry and acting was expected to excel in math because he was a male (both begin with "m"). He was shoved into commercial arithmetic.

Bertram worked two hours every night removing parentheses. Catherine worked forty minutes figuring compound interest. Andrew didn't spend any time measuring his room because the art teacher who had one math class was too busy with posters to assign any homework.

Finally the time came for assignment to tenth-grade mathematics. Bertram, who had fulfilled the high school requirements, signed up for extra sports. Catherine became convinced she had no technical future in spite of the aptitude shown by women in the war years, decided to become a dress designer and took home economics. But poor Andrew—he was a problem and a disappointment. Somehow, there must still be hope for a technical future. He was a male wasn't he? Sooner or later, with enough parental pressure and proximity to mathematics books he was sure to succeed. The triplets' junior high school mathematics counseling ends with Andrew's assignment to algebra.

Definition of the Problem

While highly imaginative, this allegory serves to highlight the major aspects of the problem from three points of view: those of the pupil, of society and of the school staff.

From the point of view of the pupil

How does the ninth-grade pupil decide what mathematics to take? It is a well-known fact that a great majority of youth are not yet ready to make final vocational choices in the junior high school. Nor

should they be. Pupils at this age should not be called on for final decisions. They are more interested in broad exploratory experiences and in general education. Yet the pupil who may follow a technical career needs four years of high school mathematics. He is forced to make a choice at grade 9: general mathematics vs. algebra. To protect his future career, the careful pupil elects a technical algebra course—just in case. Others, because this kind of choice is not required in other fields, take a course because of its prestige value, or to go along with their friends.

It follows that while the choice at the end of the eighth grade should be based on aptitude and vocational needs, sufficient program flexibility is required to allow for change of plans without penalizing the pupil.

From the viewpoint of society

One of the major purposes for mathematics teachers listed by the Post-War Commission was "to provide sound mathematical training for future leaders in science, mathematics, and other learned fields." Such leaders are bound to be lost to society unless teachers who have a major responsibility in helping to identify and direct the educational program of those youth capable of profiting from technical mathematics courses fully accept their responsibilities. The graduating physicist of 1955 was in the ninth grade in 1947. Because of his interest in science and mathematics discovered prior to the ninth grade, he was programmed into required courses at the proper time, and his career as a physicist was assured. On the other hand, far too many capable students are lost before they get started. Too frequently decisions to take or not to take algebra are based on parental aspiration or pupil whim rather than as the result of a careful study of the pupils.

If America is to meet the challenge of world leadership in what Lewis Mumford refers to as the "bio-technic age," it must develop all youth capable of maintaining scientific leadership. At the same time we are committed to the idea that each individual is free to choose his own career. This choice, to be intelligent, should be based on realistic information about the nature, requirements and rewards of the vocation as related to student's own aptitudes, abilities and interests. This means an exploration of the fields of mathematics and their vocational applications in grades 7 and 8. For the guidance counselor this means working cooperatively with the mathematics staff, the pupils, and parents, analyzing individual aptitudes and potentials as they apply to curricular plans.

From the point of view of the staff

How can courses be tailored to the needs of society and at the same time attract the students for whom they are designed? The great upsurge in pupil population, coupled with a lower drop-out rate has created a dilemma. Algebra courses tend to be watered down as they adjust to an influx of students admitted because of pupil and parent pressure. The difficult algebra courses have created a high failure rate with attendant public relations problems. General mathematics courses have become, in the eyes of the students and staff, dumping grounds for dumbbells. They are often filled with disinterested youth. The solution is being sought in multi-track programs and counseling systems. To these must be added high quality exploratory experiences, providing the data needed for guidance as well as mathematical ability needed for human affairs.

Implications for action

Any effective attack on this problem requires joint action by high school and junior high school, with special responsibilities pinpointed and accepted. This calls for administrative leadership. As Dean Cubberly so aptly pointed out, "as the principal, so goes the school." The principal with this type of orientation will be taking an active lead in reshaping the program to discover and conserve tech-

nical talent. He will probably be encouraging block scheduling to include mathematics besides arithmetic in grades seven and eight. He will be selecting new teachers who are willing and prepared to work in this program. He will encourage intelligent experimentation. To accomplish all this, he will organize and lead staff study of the scope and sequence of the traditional mathematics program with the view to making such adaptations as appear practicable in the light of individual differences and newer conceptions of the relationship of mathematics to general education. He will facilitate study and experimentation through the provision of special materials and vigorous support for those engaged in this enterprise.

It is the responsibility of the teachers through the eighth grade to prepare the pupil (and his parents) to select the appropriate sequence in mathematics. This will require:

a. Early discovery of technical competence, through arithmetic achievement, intelligence and aptitude tests, and tryout units in algebra and geometry. One school uses the Iowa Algebra Aptitude Tests, the California Mental Maturity Tests, the Stanford Binet, the Wechsler-Bellevue, achievement tests, and a special teacher evaluation for each student.

b. Exploration and discussion of the uses of mathematics in various vocations, leading to a consideration of the nature and purposes of the several sequences in mathematics in the secondary school. In Wilbur Junior High School this is handled by each teacher of an eighth-grade mathematics class. Because of the increasing concern for a more comprehensive program, plans are presently under way for better coordination and increased content.

c. A public relations program designed to secure the understanding and cooperation of the parents, directed by the administration. In one junior high school, a spring P.T.A. meeting about college entrance requirements led to the question of what mathematics courses to take. This topic was then discussed in each of the eighth-grade parent-teacher conferences.

In all of these activities, the school counselor plays an important part.

It is the responsibility of the staff beyond the ninth grade to provide a flexible multi-track program to take care of the wide range of pupil needs. Several aspects of such a plan are clearly necessary.

a. A challenging and interesting sequence in mathematics for general education. It is highly doubtful if one ninth-grade course can provide mathematical literacy for adult needs.

Such a sequence, acceptable as the normal, expected choice of the student unless he has specialized needs and aptitudes, is necessary to the solution of the present difficulty.[1] This means that the sequence must be so designed and administered that it carries prestige equal to that of other sequences.

b. A pre-technical four-year sequence, in which standards can be maintained suitable for selection and preparation of technical leaders. This means that it should be open only to those with appropriate qualifications and interests.

c. College requirements being what they are, a sequence including algebra and geometry is needed for the student preparing for a non-technical college program. Such an intermediate sequence has proved useful in providing the flexibility needed for the pupil who exercises his inalienable right to change his plans (example—tryout for 12 weeks).

It thus appears that the current difficulty which focuses on the ninth grade is an all-school problem. It has been accepted as such by the committees referred to above. What has been presented here amounts to a definition of the problem, rather than its solution. The committees can profit from experiences in other schools in medium-sized communities.

[1] Unless the general sequence is equal in quality and prestige to any other, the counseling prior to the ninth grade is futile, and the standards of the other sequences cannot be protected.

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Department of Education, Brooklyn College, Brooklyn, New York

On selling mathematical education to the public

In the pages of this department not long ago we alluded to the tenuous outlook for mathematical education. Specifically, we quoted the opinion of J. H. Murdoch to the effect that pretensions to the supreme importance of mathematics in education were currently suspect, to say the least. We should like to present Murdoch's views still further, since we feel that they are particularly well taken. He says, for example, that

> It would be silly to claim that the study of mathematics has some pre-eminent virtue satisfying its forcible imposition on all pupils. Yet it has an important place in any well-organized scheme of post-primary school education. . . .
>
> Many of the virtues commonly attributed to mathematical training are contingent only. The subject is essentially one of insight rather than of information and requires the mastery of certain specific skills. If these latter are not acquired, if the learning has been largely rote learning or uncomprehending memorization, mathematics is not being learnt at all. Conscious efforts should also be made to encourage the transfer of mathematical training to other fields of action.[1]

To us, the idea of "contingent values" attributable to the study of mathematics seems singularly appropriate and happily phrased. The same writer continues, and we take the liberty of condensing, as follows:[2]

> . . . The study of mathematics has its peculiar virtues, not so easily realizable in other fields. It has likewise its peculiar limitations and dangers, that must be recognized.

Before stating the "virtues" and "limitations," we should like to interpose our conviction that many teachers of mathematics might be well advised to reflect on the contingent nature of the virtues as well as the seriousness of the limitations when, in their enthusiasm, they are disposed to sell mathematical education to the public. (Not that we are averse to selling it—quite the contrary! But let's be realistic.) Now to state the two sides of the ledger, as Murdoch sees it. On the asset side:

> (A) Some of the beauties typical of mathematical thought can actually be enjoyed even in the third form[3]—a neat and tidy proof as contrasted with a clumsy and laboured one, a generalization that sums up with beautiful economy a multiplicity of instances, a graphical representation of functional changes enabling intermediate values to be read, the simplicity of the metric system.
>
> (B) A second feature of mathematics is the very slight demand that it makes on experience of life. . . . This freedom from experience that comes only with years is shared also by music, and it is significant that young children of genius can distinguish themselves in these two fields—music and mathematics. This has, of course, its concomitant danger, that the demands of everyday life may be ignored by those who devote themselves too narrowly to these studies. The point here at issue, however, is that mathematics deals essentially with material well within the experience of children. That teachers have managed to render the subject meaningless and obscure to so many pupils is a monument to pedagogic perversity.
>
> (C) A third feature of mathematics is that in its definiteness and precision it provides a training in exactness for the young. . . . This definiteness and precision of mathematical concepts and methods leads the pupil further to the recognition of the certainty and inevitableness of his conclusions, given his premises. He knows, or should know, without reference to his teacher, that his conclusion is correct and inescapable. This is a matter of no small moment to a young student.
>
> (D) The development of the concept of proof constitutes one of the most important aims in the teaching of mathematics. . . . It is

[1] J. H. Murdoch, *The Teaching of Mathematics in Post-Primary Schools* (New Zealand Council for Educational Research, 1950), p. 153.

[2] *Ibid.*, pp. 155–163.

[3] Corresponds to the eighth or ninth year in American public school systems.

one of the failures of our class teaching that so often we have crammed children with bookwork on the assumption that they cannot be trusted to reason for themselves.

We come now to the negative side of the ledger—the "severe limitations" of the study of mathematics. Here Murdoch is just as frank, just as sincere, and, in our opinion, just as sound as in the case of the "virtues" of the subject. Briefly, they are:

(A) Obviously [mathematics] cannot provide the literary, musical, historical, geographical and artistic training and knowledge that are provided by the other core subjects in the curriculum. Its restriction to numerical and spatial fundaments stresses an important but single aspect of life, and its predominantly intellectual character narrows still more its appeal to children. Its initial artificiality in form and symbolism is confusing, and it is so systematically developed that early failure is particularly difficult to overcome. . . .

(B) It will readily be conceded, too, that the more fundamental virtues claimed for the study of mathematics require for their realization some years of systematic study. . . .

(C) Again, there are many forms of reasoning in life that are barely represented if at all in school mathematics. This springs from the specific nature of mathematical treatment. . . .

(D) One last danger in mathematical study should be noted: it may lead to an over-exaltation of mathematical procedures and a contempt for subjects of study not amenable to its methods. Some of its leading exponents, to judge by their published opinions, are not guiltless in this respect.

O. W. Holmes and Morris R. Cohen

In the early twenties, two books which have since become rather well known were engaging the attention of lovers of mathematics: Ouspensky's *Tertium Organum*, and Spengler's *The Decline of the West*.

No small part of the interest in the former work was due to its esoteric allusions to "time-binding"—the role of time as a fourth dimension. Coming as it did at the moment when Einstein's contributions to relativity theory were uppermost in men's minds, it is little wonder that Ouspensky's mysticism caught the popular fancy. It was solemnly pronounced that "we know already by our *intellect* that *everything exists* in infinite spaces of time, nothing is made, nothing becomes, *all is*." Or, as interpreted by Claude Bragdon, who thought that these new concepts of time and space would produce a revolution in philosophical thought: "Time becomes space and space becomes a mathematical abstraction—relationships alone exist and exist perhaps only by virtue of the structure of the mind."

Interest in Spengler's morphological interpretation of history derived from his unorthodox conviction that the development of mathematics throughout the ages is *not* a continuous, cumulative evolution. "There is no mathematics, only several mathematics." According to Spengler, every culture has its own mathematic, which is inherently a necessary part of that culture; moreover, the mathematic of a particular culture is the culmination of the symbolic expression of the soul and spirit of that culture.

Among the published letters between the late Justice Oliver Wendell Holmes and his friend, the late Professor Morris Rayfield Cohen,[4] there are several letters in which these kindred spirits expressed themselves freely with regard to these two books. These letters seemed to us so intriguing that we could not resist passing them on to our readers. They are reprinted here by permission of the editors of the *Journal of the History of Ideas*, to whom we are grateful. We make no further comment on the letters, leaving them to the reader's enjoyment.

* * *

Nov. 29, 1923

Dear Cohen:

Your article[5] was duly received and read—somewhat hurriedly of necessity—but with profit and appreciation—I have more respect for the universe now that I know that there is a

[4] "The Holmes-Cohen Correspondence," edited by Felix S. Cohen, *Journal of the History of Ideas* (1948); 9: 3–52.

[5] "On the Logic of Fiction," *Jour. of Phil.*, 20 (Aug. 30, 1923), 477, subsequently appearing as chapter 5 of *A Preface to Logic.*

place in it for $\sqrt{-1}$. You are illuminating as always—and I shall try not to forget the lesson.

Ever sincerely yours,

O. W. HOLMES

* * *

June 11, 1924

Dear Cohen:

Your letter comes just as I am leaving—so I must send you only hurried thanks. You bring a sinister grin to my mug. The book[6] came too and without opening it I told my messenger to send it on by book post to Beverly Farms where I shall find it awaiting me and laboriously extract improvement from it. But I am trying to realize that it is too late to bother longer about my immortal soul and that it is lawful to seek amusement. But again I am glad to have a pièce de résistance. A dame has just sent me Ouspensky's *Tertium Organum*—with demand for an appreciation. I am suspicious—and should be glad of a hint from you as I gather from a glance that salvation lies in the fourth dimension—which is a hard look out for me. Well—I must stop. I am thankful for your friendship—My compliments to your wife whom we were very glad to meet at last.

Yours ever,

O. W. HOLMES

* * *

June 13, 1924

Dear Justice Holmes:

Teachers and judges have this in common: they must learn to read or listen to a great deal of inexcusable foolishness. I flatter myself that while my flesh is weak I have developed great patience in listening to foolish argument of students and in wading through numerous pages of nonsense to get at a possible idea or aperçu on which confused minds sometimes stumble. Ouspensky's book, however, has tried my patience beyond the three (or is it eight?) mile limit. The man has some sort of intelligence; and if he had only taken the ordinary trouble of informing himself about modern mathematics he might readily have learned how nonsensical are the things which he has put down in this *Tertium Organum*. But, alas! The charm of speculating about the incomprehensible is one of the inescapable allurements of human life.

Spengler's *Untergang des Abendlandes* is not a book for the improvement of the mind, but for lawful amusement. He has a great trick of generalizing in a way to make the facts irrelevant. But I found it very stimulating; for he opens vistas of possibility to a thinking reader who is ready to play with the author and, independently, with the subject matter.

I trust that you have now got rid of the cold and cough which you had in Washington and that you are facing the gods as erect as usual.

With kindest regards in which my wife joins,

Sincerely yours,

MORRIS R. COHEN

* * *

June 15, 1924

Dear Cohen:

Your letter greets my first morning here just as I was regretting my stupidity in leaving your last in Washington. Spengler met me on my arrival last night. You relieve my mind by what you say about him and confirm an impression from my first glance. I have read far enough in Ouspensky to believe that I shall not get much from him. He has all the earmarks of what I don't believe. He interests me mainly by recalling a talk I had with Count Schonvaloff when the Grand Duke came to Boston almost or quite before you were born—He worked off on me things that I had not heard before—the notion of a being living only on a plane—(is that from Helmholz?) and the suggestion that a point $\times$ infinity took us into a new and from the point's outlook unimaginable novelty, the line—& so the line to the plane—and the plane to the solid—whether the solid $\times \infty$ led to the 4th dimension or what, I don't remember. Of course you have got to multiply in a particular way to get the result—but it tickled me. I am hardly oriented here yet, but I had to let off a line to you and not wait to send a solid.

Yours ever,

O. W. HOLMES

* * *

June 19, 1924

Dear Cohen:

One additional word as to Spengler, to thank you and tell you how he tickles me. I read slowly as I can give only a limited time to the book and have to use the dictionary—though N.B. it is wise not to bother too much or one loses the general thought in the detail. I have read only 60 pages—but you may imagine that I chuckled at es gibt keine ewigen Wahrheiten. He gets nearer to being able to smile at himself than most Germans, though I doubt if he can—well, this is only a grunt after an hour, a happy hour, with this book—and now I must take my very modest constitutional walk—

Yours ever,

O. W. HOLMES

* * *

July 14, 1924

My dear Cohen:

This moment sees the finishing of Spengler—Damn him—he has been my task and duty since I have been here—a duty not too assiduously pursued, you can see from the time taken, even though I had constantly to turn to the dictionary. The swine has given me my money's worth—for I haven't read anything so suggestive and stimulating for a long time, from its abundant aperçus in spite of excessive repetition—I don't believe his most fundamental propositions, but

[6] Oswald Spengler, *Der Untergang des Abendlandes* (1919–1922), later translated (1926–28) as *The Decline of the West.*

I feel a lot of new light on the different *Kults* that he discusses. I infer that he is not so strong on the natural sciences as he is on mathematics, music and art—Were he not a German I should be surprised at his dogmatism in statement, when his general view is so sceptical. In spite of his scepticism he seems to feel an inward demand for absolute truth and to be disappointed at the conclusion that he can't scoop up the universe. As I read I often wished that I could consult you. I don't understand his distinction between the realms of space & cause and effect and of time and Schicksal. What is cause and effect outside of time—and what is Schicksal if not the working of cause and effect? I don't doubt that you could explain—I am perfectly willing to believe that he can't say experimentally that cause and effect are exactly equivalent —For the matter of that I have often said that if causes suddenly ceased to produce effects—or phenomena appeared without cause—and I was not too scared to think—I should simply say—Tired so soon? I thought you would last my time—But I make more modest demands of the cosmos than those who are disposed to think that it wears a beard—I might ramble on—but I just want to tell you that I have read the book—with a good deal of intellectual emotion and am deeply obliged to you for sending it to me—while my feeling toward the writer is not unmixed with malevolence—Following your intimation, which accorded with my impression from 80 pages, I have felt warranted in letting *Tertium Organum* wait for better days. Now that I have finished Spengler and sent off some accounts that are the bore of July 1, I feel the man of leisure unless you set me another task—which I shouldn't promise to perform. I hope all is well with you—my compliments to your wife whom I was so glad to meet—

Sincerely yours,
O. W. HOLMES

* * *

August 13, 1924

Dear Justice Holmes:

Your very good letter of the 14th of July has just reached me—the mail clerk at the City College kept it there four weeks.—I am naturally delighted that you liked the book and differed as much from Spengler's fundamental dogmas as I did. Spengler is a good deal of a journalist, —he is weak on the facts, in mathematics as well as in the natural sciences and also—I am informed by specialists—in art. But he has a very suggestive way of bringing together things which are not generally thought of together. He thus helps to build up new vistas or at least perspectives in which we see things in new lights. Would you like to have me send you the second volume which deals with the perspectives of world history? . . .

* * *

Just what is mathematics?

If it is somewhat difficult to tell what a mathematician is, what can be said of mathematics? From time to time, an attempt is made to describe briefly that vast enterprise collectively known as mathematics. Even a high school boy would probably agree that to describe mathematics as the "science of number, magnitude and quantity" is quite inadequate. Not much better is the claim that "mathematics is the collection of all facts resulting from logically flawless thinking." Other attempts at characterizing the Queen of the Sciences are far more telling, although each, in turn, leaves something to be desired. Thus Benjamin Peirce, a distinguished nineteenth-century American mathematician, suggested that "mathematics is the science which draws necessary conclusions." In somewhat similar vein, the philosopher William Benjamin Smith said that "mathematics is the universal art apodictic." In more recent times, Bertrand Russell's *bon mot* has become a classic: "Mathematics is the science in which one never knows what one is talking about, nor whether what one is saying is true or not." Also very suggestive is Edward Kasner's well-known quip: "Mathematics is the science which uses easy words for hard ideas." And we might appropriately bring to a close these "definitions" by recalling Cassius Jackson Keyser's observation that "the goal of mathematics is to think the rigorously thinkable."

Clearly, all such attempts to define mathematics are of little value unless one already has some reasonable notion as to what mathematics is all about. A few observations as to the scope and universality of mathematics, suggestions as to its intrinsic nature and characteristics, an indication of the attitude of contemporary mathematicians—these, perhaps, may prove more fruitful.

Our first two quotations attest to the immensity of the subject:

It would be no exaggeration, indeed, to suggest that, if any two mathematicians were chosen at random and shut up in a room, they would be so unintelligible to one another as to be reduced to talking about the weather.[7]

The mathematical universe is already so large and diversified that it is hardly possible for a single mind to grasp it, or, to put it in another way, so much energy would be needed for grasping it that there would be none left for creative research. A mathematical congress of today reminds one of the Tower of Babel, for few men can follow profitably the discussions of sections other than their own, and even there they are sometimes made to feel like strangers.[8]

What is often referred to as the universality of mathematics is briefly but aptly suggested in these passages:

Mathematical truth has validity independent of place, personality, or human authority. Mathematical relations are not established, nor can they be abrogated, by edict. The multiplication table is international and permanent, not a matter of convention nor of relying upon authority of state or church. The value of π is not amenable to human caprice. The finding of a mathematical theorem may have been a highly romantic episode in the personal life of the discoverer, but it cannot be expected of itself to reveal the race, sex or temperament of this discoverer.[9]

Mathematics is a self-determining domain of intellectual activity. It need not and does not apologize for or justify itself. If I want to know how to find the solution of an equation, or how to construct a geometrical figure, how to prove a theorem about number, or how to establish a property of space, then my only justification for wanting to know is just that I want to know. The unspoilt human mind wants to know, and recognizes no limits to the domain of this knowledge. It wants to know the ages of the pyramids, the angles of a tetrahedron, the structure of the atom, and the meaning of the square root of minus one.[10]

The essence of mathematics begins to emerge from the following considerations:

. . . In all good thinking and feeling are to be found the three great ideas underlying both logic and mathematics; viz.: *Generality; Form* (something that can be handled when its type is recognized); and *Variability*.[11]

Or, as Arnold Dresden has somewhere written:

Four aspects of the science of mathematics—its powers of analysis, its concern with functional relations, its abstract nature, and its interest in invariants under transformations . . . are all abstract in character; mathematicians cultivate them for their inherent interest. They have been turned to fruitful account probably because they reveal fundamental qualities of the human mind, but these applications belong to the sciences, natural and social.

As has often been pointed out, mathematics, more than most enterprises undertaken by man, has the distinction of being one of the most self-sufficient.

Mathematics is autonomous. What is intimate to it, its nature and structure and laws of being, must be sought in itself.[12]

Mathematics is the most completely autonomous of all human activities. It is . . . the purest of the arts, although it differs from the other arts in that the work of each period is taken up as an integral part of the work of the next, and thus comes to exist, not independently, but absorbed into a greater wholeness and beauty. Although, therefore, a history of mathematics is largely a history of discoveries which no longer exist as separate items, but are merged into some more modern generalization, these discoveries have not been forgotten or made valueless. They are not dead, but transmuted.[13]

One of the most remarkable of the intellectual achievements of man is the thoroughness with which the very foundations upon which mathematics rests have been re-examined in modern times. The movement, which began in the last decades of the nineteenth century and has not yet spent itself, soon assumed the form of an entirely new attitude toward contemporary mathematics—a viewpoint which is admirably delineated in the next half-dozen passages.

[7] S. Brodetsky, *The Meaning of Mathematics* (London: Ernest Benn, 1929), p. 3.

[8] George Sarton, *The Study of the History of Mathematics* (Cambridge, Mass.: Harvard University Press, 1936), p. 14.

[9] Progressive Education Association, report, *Mathematics in General Education* (New York: D. Appleton-Century Company, 1940), p. 256.

[10] S. Brodetsky, *op. cit.*, pp. 74–75.

[11] Jacques Barzun, *Teacher in America* (Boston: Little, Brown and Co., 1945), p. 87.

[12] R. D. Carmichael, "The Larger Human Worth of Mathematics," *Scientific Monthly* (1922), 14: 455.

[13] J. W. N. Sullivan, *The History of Mathematics in Europe* (New York: Oxford University Press, 1925), p. 10.

If my teachers had begun by telling me that mathematics was pure play with presuppositions, and wholly in the air, I might have become a good mathematician, because I am happy enough in the realm of essence.[14]

The emergence of the abstract viewpoint adopted in the Twentieth Century by a large number of mathematicians led finally to a feeling that the subject matter of mathematics was not the study of numbers or space or any elaboration thereon, but simply the determination of consequences of systems of axioms. From this standpoint any system of axioms whatsoever is fair material for investigation. Thus mathematics has come to be, at least in the eyes of many practitioners of the art, something which can be loosely described as the science of axiomatics.[15]

. . . it is surprising that a construct created by mind itself, the sequence of integers, the simplest and most diaphanous thing for the constructive mind, assumes a similar aspect of obscurity and deficiency when viewed from the axiomatic angle. But such is the fact; which casts an uncertain light upon the relationship of evidence and mathematics. In spite, or because, of our deepened critical insight we are today less sure than at any previous time of the ultimate foundations on which mathematics rests.[16]

The ultimate foundations and the ultimate meaning of mathematics remains an open problem; we do not know in what direction it will find its solution, nor even whether a final objective answer can be expected at all.[17]

[14] George Santayana, *Persons and Places* (New York: Charles Scribner's Sons, 1944).

[15] R. B. Kershner and L. R. Wilcox, *The Anatomy of Mathematics* (New York: Ronald Press Co., 1950), p. 27.

[16] Hermann Weyl, "The Mathematical Way of Thinking," *Science* (1940), 92: 446.

[17] Hermann Weyl, *The Philosophy of Mathematics and Natural Science* (Princeton, N. J.: Princeton University Press, 1949), p. 219.

. . . mathematics is eminently interested in the methods by which concepts are defined in terms of others and statements are *inferred* from others.[18]

. . . mathematics, like other cultural entities, is what it is as a result of collective human effort directed along evolutionary and diffusionary lines. And what it becomes will not be determined by the discovery of "mathematical truth" now hidden from us, but by what mankind, via cultural paths, makes it.

In short, mathematics is what we make it; not by each of us acting without due regard for what constitutes mathematics in our culture, but by seeking to build up new theories in the light of the old, and to solve outstanding problems generally recognized as valuable for the progress of mathematics as we know it. Until we make it, it fails to "exist." And, having been made, it may at some future time even fail to be "mathematics" any longer.[19]

And so, in leaving the question: what *is* mathematics? we concur with the observation of Professor S. T. Sanders, who pointed out that

the non-existence of a universally received definition of mathematics is regarded by *no one* as a blemish on the science. On the contrary, the fact that the body of mathematical knowledge today is vastly extended beyond those lines that marked its boundaries even as late as a century ago, lends strength to the idea that the continued lack, from age to age, of a widely accepted definition of mathematics, has been fortunate, rather than unfortunate for the expansion of the science.[20]

[18] *Ibid.*, p. 3.

[19] Raymond Wilder, *Introduction to the Foundations of Mathematics* (New York: John Wiley & Sons, 1952), pp. 283–284.

[20] S. T. Sanders, *National Mathematics Magazine* (1945), 19: 326.

"If we work upon marble, it will perish,
If we work upon brass, time will efface it.
If we rear temples, they will crumble to dust.
But if we work upon men's immortal minds,
If we imbue them with principles,
We engrave on these tablets something which
 no time can efface,
And which will brighten and heighten to all
 eternity."

—*Daniel Webster*

A column of unofficial comment

The International Mathematics Union, The International Congress of Mathematicians, and The International Commission of Mathematical Instruction

by Howard F. Fehr, Teachers College, Columbia University, New York, New York

The mathematical societies and associations of more than forty countries of the world are united by an over-all organization called the International Mathematics Union (I.M.U.). The Union has its own set of officers and executive board and is supported by contributions by the member societies and other philanthropic institutions. Its main purpose is the communication of mathematics research among all its members. To expedite this purpose the Union arranges a meeting every four years to which mathematicians from all parts of the world are invited to attend. This meeting is called the International Congress of Mathematicians.

The last Congress was held at Amsterdam, The Netherlands, from September 2 to September 9, 1954, and was attended by more than 1700 delegates, including research mathematicians, university and college professors, high school teachers, industrial mathematicians, and statisticians. Specialists in the various areas of mathematics study were invited to present hour and half-hour lectures and contributing papers of fifteen minutes in length. Seven sections were provided for special interests with the titles: (1) Algebra and Theory of Numbers, (2) Analysis, (3) Geometry and Topology, (4) Probability and Statistics, (5) Mathematical Physics and Applied Mathematics, (6) Logic and Foundations, and (7) Philosophy, History, and Education. The summaries of the short addresses have been published in Volume 2 of the *Proceedings of the International Congress of Mathematicians* available at P. Noordhoff Ltd., Groningen, Holland.

The members of the National Council will be most interested in the presentations in the sessions of Section 7, which were arranged and presided over by Dr. L. H. N. Bunt, professor at the University of Utrecht. There were six sessions of this section at which 36 addresses were presented, 5 different languages were used, and 12 countries were represented. The range of interest can be best estimated by the following sampling of titles of addresses: "School Mathematics for Life"; "Vector Analysis in School Mathematics"; "Projective Thinking"; "The Mathematics Laboratory"; "Intuitive Methods"; "Intuition and Rigor in Instruction"; "Didactical Research in Mathematics Education"; "Geometry and Reality"; "Learning Through Insight"; "The Notion of Structure in Elementary Mathematics." Another notable feature of this section was an exposition of secondary school mathematics textbooks and periodicals from the major countries of Europe

and from the United States. This exposition has been preserved in Paris and is available for display in other countries. A comparison of textbooks is highly illuminating of different philosophies of mathematics instruction.

One of the sessions, sponsored by the International Commission on Mathematics Instruction, was devoted to "Mathematical Instruction for Students Between 16 and 21 Years of Age." Descriptions of this instruction and the trend were given by Dr. H. Behnke (Germany); A. Chatelet (France); S. MacLane (United States); M. Villa (Italy); Miss M. L. Cartweight (England); L. H. N. Bunt (Holland); O. Frostmann (Sweden); F. Hohenberg (Austria); and S. Bundgaard (Denmark). It is possible here to cite only the most interesting programs and trends. In Germany the program has been simplified by removing that subject matter of only historic value, and replacing it by instruction that leads more rapidly to modern mathematics—algebra of sets, statistics, topology. This can be done because it is not true that all modern mathematics is more difficult than the traditional mathematics, but it is frequently simpler, more interesting, and leads directly to use. In most countries there was expressed consideration for the learner, and a replacement of purely lecture and study methods by so-called methods of rediscovery through problem solving and the use of intuition.

The report for the United States, given by Saunders MacLane of the University of Chicago, was excellent both in its objectiveness and clarity. Among other things, it pointed out a trend toward fewer hours of school study devoted to mathematics, a smaller percent of students studying mathematics in high school and college, and the effect this would have in the future supply of scientific manpower.

A second session of the Commission was given over to reports on "The Role of Mathematics and Mathematicians in the Present Age." Professor Kurepa of Yugoslavia made an extended survey of these roles in all the countries of the world. His report, when completed, will be published in *L'Enseignement Mathématique,* Genève. His preliminary report contained much material for teachers to use in showing the great force mathematics plays in modern culture.

Of utmost importance to teachers of secondary school and college mathematics is the nature and function of the International Commission of Mathematical Instruction (I.M.C.I.) (in Germany called the *Internationale Mathematische Unterrichtkommission* (IMUK), and in France called *Commission Internationale de L'Enseignement Mathématique*). This Commission was first organized in 1908, then reorganized by the Mathematics Union at a special meeting in Rome, 1952. At the Amsterdam Congress a permanent organization of this Commission was established and officers and representatives appointed to serve from January 1, 1955, to December 31, 1958. The organization consists of a president and an executive board of ten members, appointed by the Union to serve terms of four years. (The president may be included as one of the ten board members.) Each country represented in the Union is entitled to two representatives in the Commission. From the list of representatives, and the appointed executive board, there shall be elected two vice-presidents, a secretary and a treasurer to serve as officers on the executive board. (If these *officers* are selected from other than the ten appointed board members, the board becomes enlarged in membership.) The Commission thus consists of the officers, members of the executive board, and the appointed representatives from the various countries. This group decides policy and program.

In addition, each country in the Union may appoint a sub-committee of the Commission for that country. In the United States the sub-committee will most likely be named by the Mathematics Society and the Mathematics Association, in consultation with the National Council of Teach-

ers of Mathematics. The sub-committee can be of any size, but two members of this sub-committee must be that country's representatives to the Commission. It has been recommended that each country, in selecting its two representatives to the Commission, select a mathematician with interest in pure or applied research and another mathematician with a large concern for secondary and collegiate instructional problems in mathematics education.

The ten members appointed to the Commission by the Union, to serve from January 1, 1955, to December 31, 1958, are: H. Behnke, President, Munster, Germany; Y. Akizuki, Japan; G. Ascoli, Italy; P. J. Dubreil, France; J. C. H. Gerretsen, Netherlands; R. L. Jeffery, Canada; D. Kurepa, Yugoslavia; E. A. Maxwell, England; Ram Behari, India; and M. F. Stone, United States. All these men are pure mathematicians of high rank and respected for their interest in mathematics education either at the secondary or collegiate level. However, no one in this group can be considered a specialist in educational problems in secondary and college mathematics.

The following projects have been proposed for investigation by the Commission during the years 1955–1956: (1) producing treatises on the foundations of modern mathematics for secondary school and junior college teachers to aid them in properly teaching mathematics which is genuinely *modern;* (2) a continued study in every country of the present and possible changes in mathematics education of youth, ages 16 years to 21 years; and (3) a continued study of the role that mathematicians and mathematics play in our world today.

The purpose of the International Commission on Mathematics Instruction, the problems it is studying, and the necessity that educators and mathematicians work cooperatively make it important that The National Council of Teachers of Mathematics keep well informed of, and take an active interest in, the work of the Commission. The forthcoming reports of this Commission should be made known to and interpreted by all the mathematics teachers of the United States, if we are to make progress and to maintain our rightful educational place in the community of nations.

The organization of the International Congress of Amsterdam was superb. Mathematicians of every country of the world, regardless of color, race, religion, or political belief, were received with dignity and given the hospitality of a city that will long be remembered by every delegate. The Organization Committee and the people of Holland have the deep gratitude of the mathematicians from all over the world.

"Scientific progress is like mounting a ladder; each step upward is followed by a brief pause while the body regains its balance, and we can no more disregard the steps which have gone before than we could cut away the lower part of the ladder."—*O. G. Sutton,* Mathematics in Action (*London: G. Bell and Sons, 1954*)

Reviews and evaluations

Edited by Richard D. Crumley, University of South Carolina, Columbia, South Carolina, and Roderick C. McLennan, Arlington Heights Township High School, Arlington Heights, Illinois

BOOKS

First Course in Calculus, Hollis R. Cooley, New York, John Wiley & Sons, Inc., 1954. Cloth, v+643 pp., $6.

In the preface the author states: "The book is to be addressed to the students. It is not a mere exercise book; it is to be read. . . ." The author has succeeded in his intention. In the approach to new topics the author has taken pains to develop each new concept. Rigor has not been sacrificed, but neither has it been made a fetish. The book has been written for beginning calculus students to read and understand.

Somewhat different from most other texts is the manner in which integration is introduced. In Chapter 6 the topic is introduced as inverse differentiation. The symbol D^{-1} is used. The usual integral sign and the term "indefinite integral" appear two chapters later after the definite integral has been introduced as the limit of a sum. This change from tradition will probably receive both praise and condemnation. The change should be tried before an opinion is formed.

The sequence of topics is that found in most of the texts published in America in the past two decades. Differentiation of algebraic functions is followed by inverse differentiation (integration), definite integrals, and applications, before the transcendental functions are introduced. Trigonometric functions are introduced in Chapter 10 and both differentiation and integration are treated. The same procedure is followed in Chapter 18 with the hyperbolic functions.

Chapter 23 contains an excellent treatment of the essentials of solid analytic geometry. This is followed by chapters on partial differentiation and on multiple and iterated integrals.

Throughout the book there are excellent lists of story problems, which do much to help the student to realize the power and importance of the methods of calculus. There are also satisfactory lists of examples for drill purposes in developing essential techniques. However, the emphasis seems to be on the applications, as it should be.

Professor Cooley has made a real contribution to the list of American texts in calculus.—*P. D. Edwards, Ball State Teachers College, Muncie, Indiana.*

Introductory College Mathematics, Adele Leonhardy, New York, John Wiley & Sons, Inc., 1954. Cloth, v+459 pp., $4.90.

This text deals with a wide variety of mathematical topics. For each topic the author provides good illustrative exercises, then exercises for the reader to work. These exercises are interesting. I believe most readers will feel that they are practical. So I find the text best adapted to teaching pupils how to solve many types of mathematical exercises.

Chapter-by-chapter the author inserts historical notes that will interest many readers. The pupil who studies these notes carefully will acquire a fund of information about the development of mathematics.

The author tries to introduce the reader to the nature and method of mathematics. In this she is less successful. I believe there are two somewhat promising ways to accomplish such an introduction:

One is to write what mathematicians call *elegant* proofs and expositions. Statements like: "A zero of a function is any value . . . ," page 202 (Why *any?*), and "Two variable sets vary directly if the quotient of their corresponding number pairs is constant," page 229 (What are variable sets?) seem to me to lack the elegance of the best contemporary writing in the field of mathematics. Moreover, the distinction between what's proved and what is made to seem reasonable is not sharply made. For example the *proof* that $b^m \cdot b^n = b^{m+n}$ on page 89 appears to be a formal, rigorous proof. Yet modern standards of rigor demand a proof by mathematical induction. On page 57, the author *demonstrates* that $(-a)(-b) = +ab$. She does this indirectly by showing that the answer $-ab$, "*the only other possibility*," leads to a contradiction. She says this is not really a proof but "is intended to show that this 'rule of the game' is consistent with our previous assumptions"; but, to the reader, it has the appearance of a proof as presented. The handling of "proofs" in this text seems to me a confusing mixture of rigor and plausibility.

The other promising approach, as I see it, is to lead a reader to generalize his ideas and formulate interesting abstractions. In this text the characteristic order of presentation is: pep talk on the importance of a topic; rules and new words; tricks you can perform by following the

rules. The pupil is not a partner in discovery. He is shown the finished product.

The reader who is docile and persistent will learn many tricks of mathematics. I do not believe he will learn mathematical method from this text. He may even develop hostile attitudes as he grapples with quasi rigor and fights his way through tricky sentences for whose subtleties he is conceptually unprepared.—*H. C. Trimble, Iowa State Teachers College, Cedar Falls, Iowa.*

Basic Ideas of Mathematics, Francis G. Lankford, Jr., and John R. Clark, New York, World Book Company, 1953. Cloth, vii +504 pp., $2.84.

Mathematics for the Consumer (Revised), Francis G. Lankford, Jr., Raleigh Schorling, and John R. Clark, New York, World Book Company, 1953. Cloth, 438 pp., $2.76.

Although either may be used separately, these books could well be used in sequence in a secondary school general mathematics course having as its objective the teaching of number concepts in a very practical, rather than theoretical, way.

From a review of elementary school arithmetic, the "basic" book goes into a study of elementary geometry, and by the middle of the book, algebra has been introduced. The latter part of the book emphasizes practical applications of fundamental algebraic processes. The first part of the book provides an excellent transition from grade school arithmetic to secondary school mathematics, but the last half of the book moves perhaps too rapidly into what would be for some students the "unknowns" of algebra.

Mathematics for the Consumer contains a considerable amount of good descriptive material concerning business transactions that the average person will frequently come in contact with, and it would therefore be a useful teaching aid for teachers who may not be too familiar with typical business processes. There are other somewhat similar books available that contain more mathematics problems, but it is doubtful if the mere working of long series of similar problems would be very valuable in a course whose primary purpose would be to help the mature student learn to think out for himself the financial problems he will face after he leaves school. A terminal course in practical mathematics should feature class discussions, and this book is filled with material that would help to promote this technique.

The idea of presenting mathematical material in such a way that it can be grasped by almost every high school student and then going on to demonstrate the practical uses to which these concepts can be put is very commendable. It is generally recognized by secondary school mathematics teachers that higher institutions of learning must eventually be persuaded to accept, with a greater degree of latitude than at present, the idea that many mathematical concepts can be grasped by some students in ways other than those dictated by tradition. Unfortunately many colleges are as yet unwilling to substitute high school credits obtained in such courses for the traditional algebra and geometry material. A school using these texts would therefore have to be very careful to avoid the possibility of a student completing the courses and then being disappointed because a certain college would not accept the credit.—*John W. Rau, New Trier Township High School, Winnetka, Illinois.*

What Is Science? Norman Campbell, New York, Dover Publications, Inc., 1952. 186 pp., Paper-bound: $1.25; Cloth-bound: $2.50.

The title and purpose of this book are intriguing; it does not live up to the expectation. It is implied that the intention is to write an introduction to science for the general public and to encourage them to go on to study more of it in detail.

The central idea of the book is the "laws of science." The processes by which observation of natural phenomena are inductively combined into laws is told in a leisurely and friendly fashion. The examples drawn from science are not too advanced and are close enough to common experience to appeal to most people.

At present the book is being reviewed from the standpoint of one who is interested in mathematics and its relationship to the other sciences discussed. The artificial and constant avoidance of any mathematical symbolism or simplification makes explanation *more* complicated rather than *less* and gives the false impression that science records and transmits its ideas in long, wordy arguments rather than the direct, terse algebra which is one of the jewels of mathematics, even at an elementary level. This is particularly troublesome in the chapters on "Measurement" and "Numerical Laws and the Use of Mathematics in Science." The picture is not correct.

Scientists may agree with the constantly reiterated statement that science deals with "laws that are capable of universal agreement." The type of finality, eternal acceptance, and rock-bottom truth which must be attached to all "laws," and therefore all real science, bothers anyone with even a little understanding of the postulational method in mathematics. Some of the most fruitful investigations in physics (the author's field) and in mathematics have been concerned with doubtful premises, partially accepted "laws" and intuitive or even mystical hypotheses.

A fundamental fault is the attempt to cover a great deal, with much depth of inquiry, and always to say it in simple language. The material which the book tries to cover is important and

fundamental, the author doubtless has the background to cover it, but he repeats so often and strings out his explanations so much that one has the feeling of wandering in an ill-arranged museum. Oh, for a clear floor plan!

It is possible to find contradictions without too much trouble. On page 179 we find that "Science brings to the analysis of . . . experience the concept of definite, positive and fixed laws"; yet page 182 says, "Applied science, like pure science, is not a set of immutable principles and propositions; it is rather an instrument of thought and a way of thinking." One does not get caught at this in lectures; but a book is different. There are better books on this same subject.—*Henry W. Syer, School of Education, Boston University, Boston, Massachusetts.*

Reviews and Examinations in Algebra, Oswald Tower and Winfield M. Sides, Boston, D. C. Heath and Co., 1935. Cloth, v+183 pp., $2.28.

This book contains a large amount of supplementary problem material, especially for the second year of the intermediate course. Problem lists for the various topics begin at the level of difficulty of conventional texts but include a great many more challenging exercises. The latter half of the book is composed of a variety of examinations including college entrance and "prize" or challenge exams from the files of several well-known private schools. No answers are included, which some users will regret.

There is increasing interest among educators in providing stimulating work for the superior student. This book can be warmly recommended for this purpose. However, because the reviewer believes it will be widely used in this manner, some suggestions are offered, if and when a revision is contemplated. There should be more use of the functional notation and in general a broader treatment of the function concept. More emphasis might be given to relatively easy concepts approached in a variety of ways, rather than on problems of greater technical difficulty always stated in conventional phrasing. When extending the range of second-year topics, analytic geometry would be more useful than theory of equations. However, as is, a room set of this text would be valuable to every school offering second-year algebra.—*L. Clark Lay, John Muir College, Pasadena, California.*

Induction and Analogy in Mathematics, Vol. I, and *Patterns of Plausible Inference*, Vol. II, of *Mathematics and Plausible Reasoning*, G. Polya, Princeton, New Jersey, Princeton University Press, 1954. Cloth, xvi+280 pp., $5.50 (Vol. I); x+190 pp., $4.50 (Vol. II); the set $9.00.

There have been frequent criticisms that most mathematics texts present only polished proofs without sufficient indications of the steps used to discover the proofs. These two books by Professor Polya are primarily concerned with the reasoning (plausible reasoning) used to discover properties and theorems, to make intelligent conjectures, and to test these conjectures. Plausible reasoning may be based upon induction, analogy, and subtle inferences of many kinds. Unlike demonstrative reasoning with its formal proofs of conclusions that are as true as the hypotheses, plausible reasoning is used to strengthen or weaken one's belief in a conjecture. In other words, plausible reasoning may be compared to a force with direction and magnitude. The direction (supporting or discrediting the conjecture) may be logically established; the magnitude of the force (significance of the reasoning) is usually subject to personal interpretation.

The books should be enjoyed by readers with a thorough knowledge of elementary algebra and geometry, some knowledge of analytic geometry and calculus, and some past experience with limits and infinite series. There are also many examples and comments that will be appreciated only by readers with more advanced training, but these occasional advanced concepts should not disturb the continuity of the books for readers with the above minimum of mathematical background.

Professor Polya presents a forceful argument for the teaching of intelligent *guessing* as well as *proving*. In this sense plausible reasoning is concerned with the distinction of "a more reasonable guess from a less reasonable guess." The guesses facilitate the proofs and the proofs are necessary to establish the guesses.

The reviewer's enjoyment of the books was greatly enhanced by the wide variety of problems and problem situations—most of them elementary, many with a human interest rather than a mathematical orientation. There are also very readable and enjoyable discussions of such concepts as the isoperimetric problem (Chapter X, Vol. I) and "chance, the ever-present rival of conjecture" (Chapter XIV, Vol. II). The reviewer's major disappointments were in the absence of an index ("since an index would render the terminology more rigid than it is desirable in this kind of work") and the emphasis upon using all of the given data. Even though the latter is an important aspect of formal mathematical problems, it is a handicap in applied mathematics where the precise formulation of the problem and the determination of the type of data needed form a basic part of the problem. —*Bruce E. Meserve, New Jersey State Teachers College, Montclair, New Jersey.*

EQUIPMENT

Slated Globes, Denoyer-Geppert Company, 5235 Ravenswood Avenue, Chicago 40, Illinois. Slated globes in wood cradles; available in 6", 8", 12", 16", 20", and 24" sizes;

8″ globe, $4.25 plus postage; 12″ globe, $22.00 plus express charges.

The two globes examined were 8″ and 12″ in diameter and finished in black. The 8″ globe is constructed of metal, and the 12″ globe is made of molded wood-fiber plastic. Both globes come with wood cradles. The metal globe has a seam around the equator and rotates in any direction in the cradle. The 12″ globe is seamless and can also rotate in any direction or, by using the metal rod furnished, can rotate about an axis of $23\frac{1}{2}°$ from vertical.

The quality of workmanship represented by these globes is excellent. The 12″ model is hand-assembled and hand-finished, and this accounts for the higher price of it. Any of these globes can be used effectively in instruction in the areas of solid geometry, spherical trigonometry, navigation, or intuitive geometry. The producer recommends using a very soft chalk, and wisely so, because hard chalk tends to scratch the surface. Yellow, soft chalk was found to show up well and to be easily removable by using a soft, dry cloth followed by a damp cloth. Paper cleansing tissue was also found to be effective in cleaning the globes. Dropping the globes onto a concrete floor from a height of two feet did not seriously damage either globe. The metal globe came apart at the seam, but it was reassembled without much effort. The 12″ globe did not dent or crack from the fall, but the surface did wrinkle a little. Reasonable care of these globes should insure a long period of utility.—*Richard D. Crumley.*

FILM

Story of Weights and Measures, Coronet Instructional Films, Coronet Building, Chicago 1, Illinois. 16 mm. film with teacher's guide; collaborator, Foster E. Grossnickle; 11 min.; b & w, $55; color, $100.00.

This film presents a historical development of measuring devices for measuring length, area, and volume, with emphasis on the need for standard measures. Some of the early procedures and devices for measuring are shown in their original setting. The producer recommends that the film be used in grades 4–9.

This is an interesting film and well suited for grades 4–9, especially for grades 4–7. Because of its explanation of the evolution of measuring devices and units, it should serve to help develop interest in learning about units of measure and their use. The medium of film is used to advantage by presenting things impossible or impractical to be presented in the classroom. The technical quality of the photography and sound is excellent.—*Richard D. Crumley.*

PAMPHLETS

A Discussion of Family Money: How Budgets Work and What They Do, Women's Division, Institute of Life Insurance, 488 Madison Avenue, New York 22, New York. Pamphlet, $8\frac{1}{2}″\times11″$, 24 pp., free.

Description: This pamphlet presents an informal report on the principles of good family money management. It includes a survey of the necessary procedures involved in planning the use of the family's money for the benefit of all concerned. Sections are devoted to planning the use of family money, starting out to manage your money, the family income, and keeping down living expenses. The pamphlet does not attempt to develop a "standard budget" or convince the reader that a family should confine its expenditures to "average" amounts spent by others. The emphasis is on developing a plan through family co-operation which will lead to individual and family self-satisfaction and happiness. The pamphlet is devoid of mathematical content except for a family financial plan and some graphs summarizing a family situation upon death and retirement.

Evaluation: Although the amount of mathematical material in this pamphlet is limited, it would be very useful for supplementary reading during a unit on budgeting for junior or senior high-school students. The pamphlet is clearly and concisely written, employing a very reasonable point of view in regard to a sound system of money management.—*Clarence Olander, St. Louis Park, Minnesota.*

What Life Insurance Means, Educational Division, Institute of Life Insurance, 488 Madison Avenue, New York 22, New York. Pamphlet, $6″\times9″$, 24 pp., free.

Description: This pamphlet furnishes concise information on the nature and purposes of life insurance. Insurance is described as a means by which people share risks. Death is portrayed as the greatest risk of all and hence the need for life insurance. Topics such as living and death benefits, life insurance investments, insurance programming, types of insurance policies, and options regarding insurance benefits are well treated in this publication. Convenient summaries in chart form of much of the information are included. Incompleted cash value and mortality tables are included, but very little is mentioned of their function or use. Emphasis is placed on careful purchasing of life insurance to meet the individual needs of a particular family.

Evaluation: This pamphlet could well serve as the basic reference for a unit on life insurance. However, it must be emphasized that the mathematical content is restricted to the reading of a few tables and charts. Nothing is said of the role of mathematics in determining insurance premiums or in computing premiums for policies of varying types and face values. The pamphlet is well prepared and includes sufficient information to provide a high-school student with an adequate background concerning life insurance.—*Clarence Olander, St. Louis Park, Minnesota.*

Edited by Francis G. Lankford, Jr., University of Virginia, Charlottesville, Virginia

Building an appreciation of mathematics

by Elizabeth M. Alley, O'Henry Junior High School, Austin, Texas

There is ample reason why students should take from their study of mathematics pleasant memories, a sincere appreciation of the beauty and worth of the subject, and an interest that may enrich their leisure and provide recreation in the years ahead of them. For some, mathematics may become a means by which the mysteries and beauties of the universe are unfolded. To build an appreciation of mathematics is an appropriate goal in the early study of arithmetic as well as in the study of later courses. However, it is my experience that units planned primarily to develop an appreciation of mathematics should not be *required*. Moreover, conventional tests are not suitable to measure the development of such an appreciation. The teacher must informally observe pupils' reactions to determine their attitudes toward mathematics. To the uninitiated, mathematics is generally considered a purely unemotional subject dealing with the practical and the coolly calculating aspects of life. But the person who has studied mathematics with success and enjoyment has experienced many highly satisfying emotions.

There are many pupil activities that may be used to develop an appreciation of the beauty and worth of mathematics. Pupils may collect examples of unusual as well as everyday uses of simple mathematics and display them on the bulletin board. They may write and produce dramatic skits based on highlights in the history of mathematics, or they may write essays containing such content as the following.

Mathematics everywhere

Mathematics may appeal to us through our eyes and ears. Whoever thinks to connect music with mathematics? Yet music, like other forms of art, has a mathematical foundation. Music is divided into mathematical units of notes and measures. In "The March of the Wooden Soldiers," there are four quarter notes, or their equivalent, to every measure. This arithmetical division of note values makes march time. If there are only three quarter notes to the measure, we have waltz time as in the "Blue Danube Waltz."

Through music, mathematics appeals to the ear; in architecture, its appeal is to the eye. The pyramids, those magnificent tombs of the ancient Pharaohs, are built on square bases with sides of four equilateral triangles coming to a common vertex. The Parthenon, great ancient temple of Athens, is a perfect rectangle with seventeen parallel columns on the side. Its dimensions are 228 feet by 101 feet by 65 feet. In contrast to the parallel lines of the Parthenon, the Tower of Pisa presents an example of oblique lines. Its general form is that of a cylinder. The Coliseum is a perfect circle, and the dome of the Taj Mahal is a semisphere. In our own time and country, we find in Washington, D. C., a famous building with five equal sides. Distinctly a mathematical figure, it is called by a mathematical name—the Pentagon.

Architecture is only one of the arts that makes an appealing use of mathematics. Painting and sculpturing depend on mathematics for ratio, proportion, symmetry, similarity and congruence. The human body represents some of these mathematical properties.

Next we may consider the mathematical plan of the universe. The underlying properties of much of the beauty and grandeur of nature are mathematical. The earth is a sphere turning with absolute mathematical precision on its axis. Look around you. The horizon is a circle. The sun, the moon, the stars, the rainbow are all mathematical forms which we study. Within the hexagonal snowflake are myriads of regular polygons whose indescribable beauty is so minute that it can be seen only through the microscope.

Lilies, narcissus and jonquils are exquisite examples of the hexagon. The cosmos is an octagon. The leaves of the poplar tree, the maple, the cottonwood, and the ivy vine are multicolored illustrations of similar triangles.

Though we have been talking about mathematics as it is manifested in the greatness and beauty of art, it is one of the most practical and utilitarian subjects studied. From the classes in homemaking come a round doughnut, star-shaped gelatin salad, tea in a cylindrically-shaped glass, an oval spoon, a round bowl filled with ice cubes, and a wedge of cake. Think of those who cook. Fractions, fractions! We have to measure, multiply, divide, add and subtract. What is a third of three-fifths of a cup of sugar? We even *eat* mathematics!

The world we live in—our everyday lives, and our business activities, as well as our cultural pursuits are based largely on the simple principles of addition, subtraction, multiplication, and division. That is why we include such units as taxation, banking, insurance and measurements in our study of general mathematics. These practical applications of mathematics are an important part of the training for becoming effective citizens of tomorrow.

REAL GEORGE **By Les Landin**

"I got suspicious when she started putting her unsolved arithmetic problems in her love notes. . . ."

Used by permission of the **NEA Journal**

• WHAT IS GOING ON IN YOUR SCHOOL?

Edited by John A. Brown, University of Wisconsin, Madison 6, Wisconsin, and Houston T. Karnes, Louisiana State University, Baton Rouge 3, Louisiana

Special preparation for general mathematics teachers[1]

by Lee E. Boyer, State Teachers College, Millersville, Pennsylvania

The widespread recognition of two "tracks" in secondary school mathematics education leads to the issue implied in the title. The matter is particularly noteworthy because in the near future the schools of our country will be offering more classes of "general" ("basic," in Pennsylvania and California; "functional," in Florida) mathematics than of "traditional" ("specialized," in Pennsylvania; "traditional," in Florida; "college preparatory," in California) mathematics.

Believing that beginning teachers teach pretty much as they were taught, we at Millersville asked ourselves, "Is the conventional program of algebra, trigonometry, analytic geometry, calculus and theory of equations, as we are accustomed to teach it to prospective mathematics teachers who themselves specialized in mathematics in high school, the best way of preparing general mathematics teachers? Are the habits of work, attitudes toward the subject, depth of abstract comprehension, and rates of subject coverage emphasized in such classes similar to that which these teachers will find in their general mathematics classes? If not, might this program be, at least in part, responsible for the all-too-frequent willingness of teachers (and, in turn, administrators and pupils) to stigmatize the general mathematics courses and their pupils?"

Our first step then, as an institution, as a department, and as teachers, is to recognize somewhat realistically individual differences among students and between mathematics courses organized to meet different needs. We believe that, at present, it is probably more difficult to teach general mathematics than to teach traditional mathematics. It is therefore necessary for our prospective mathematics teachers also to sense the problem of teaching general mathematics successfully and, as they consider their developmental program, to study their own teaching powers and equipment so they will be especially well prepared either for the job of teaching the traditional mathematics or the general mathematics. By thus focusing attention on the teaching of general mathematics we attempt to get prospective general mathematics teachers to believe, as a result of actual experiences, that the general mathematics courses are some of the most worthwhile courses in the entire high school's course offerings.

To date we have been caring for our prospective general mathematics teachers, in the main, with our prospective traditional mathematics teachers. However, in the courses "Teaching of Secondary Mathematics" and "History of Mathematics" we lean over backwards to emphasize the general mathematics pro-

[1]*Editorial note:* For a more general discussion of this subject see Lee E. Boyer, "A New Responsibility of Teacher Education Programs," The Mathematics Teacher, XLVII (February, 1954), 66–70.

gram. Our course, "Field Work in Mathematics," is literally filled with opportunities to see practical applications of mathematics and to use many kinds of instruments and devices in connection with mathematical problems. The teacher of this course, Associate Professor George R. Anderson, is especially gifted to inspire students to build mechanical aids of a mathematical nature.

To date our course called "Fundamentals of Mathematics," covering pretty largely the content of the author's textbook,[2] is not offered to students minoring or majoring in mathematics. It is required of all prospective secondary teachers in fields other than mathematics and of the prospective elementary teachers planning to teach in grades four to six inclusive. However, the members of the mathematics department offering the course have, upon several occasions, suggested that we should attempt to have something similar to it available to those prospective mathematics teachers who plan to specialize in the teaching of general mathematics.

As for methodology of teaching we like to point out, by actually doing certain units, that the methods of general mathematics are somewhat more laboratory like—not that textbooks are not studied—but the spirit of study and classwork is more like that of conducting practical investigations. Although the teacher always has his eye fixed on promoting growth of mathematical power, he relies heavily on enthusiastically using many life applications of mathematics to win the student to believe that there really is merit in learning about and effectively using the established methods and techniques of mathematics. The problem resources of the local community are embraced—this includes newspapers, magazines, journals, and radio and television programs. Intergroup and group relationships are relied upon to add to the opportunities to learn by doing. The primary aim of promoting the all-round development of the student through his study of general mathematics guides the teacher-student relationship in college as it should guide the same relationship in high school.

Perhaps our effort could be best characterized by saying that we merely make a special effort to help our prospective mathematics teachers transfer the rather standardized mathematics teaching task from teaching a group of pupils with a relatively narrow range of interests to teaching one with a much wider range of interests.

[2] Lee E. Boyer, *An Introduction to Mathematics for Teachers* (New York: Henry Holt & Company, Inc., 1945).

Mathematics made meaningful in teaching graphs[1]

by George Janicki, Elm School, Elmwood Park, Illinois

In teaching the graphing unit in seventh- and eighth-grade mathematics, I have tried to give meaning to the facts being presented by using new ideas which would be appealing and therefore be more interesting.

I found my best source to be the *Information Please Almanac*, by John Kieran, which certainly contains sundry statistics. Also found useful were science books and other supplementary arithmetic books.

After introducing angle measurement and the use of the protractor, and division of a circle into certain parts according to

[1] Talk given on the Mathematics Panel, on Leyden Institute Day, March 12, 1954, at Leyden High School.

the related percentages, I proceed to teach the Circle Graph Unit as follows:

1. How We Learn: 87% by Our Eyes, 13% by Our Ears.

Then I decided to use the central idea of favorites, of favorite things to do, to see, to eat, to wear and to enjoy.

2. *Favorite TV Programs:* Comedies 55%, Mysteries 30%, Music 10%, News 5%.
3. *Favorite Types of Music:* Classical 30%, Semi-Classical 20%, Popular 50%.
4. *Favorite Type Books:* Mysteries 25%, Adventures 65%, Sports 10%.
5. *Favorite Amusements:* Sports 30%, Reading 25%, Movies 25%, TV 20%.

From the field of science, I found some interesting data on the topic of Human Blood Types: Type "A" 41%, Type "B" 10%, Type "O" 45%, and Type "AB" 4%.

Also somewhat useful is the Water Composition as given by the formula:

H_2O or Hydrogen $66\frac{2}{3}\%$ and Oxygen $33\frac{1}{3}\%$.

From the field of language study, the topic of English Language Sources was found to be appealing: how our language grew from the Anglo-Saxon and the Teutonic tongues.

When considering vertical bar graphs, there were found some useful suggestions in *Thinking with Numbers.*[2]

I used the following topics with appropriate measurements and had many ideas based upon them continued in other graphs. Some students even wanted to make up their own graph material and presented it to the classes.

Science facts—anything to do with planets and their measurements, their distances from the sun, the length of their orbits, their relative sizes.

Longevity of certain animals—including man—proved to be humorous and interesting. The weights of certain breeds of dogs were considered by many students to be untrue unless their dog's weight was considered in the data.

In teaching safety, I use the topic of the braking distance of cars traveling at certain speeds to be an effective way of presenting the idea.

Famous buildings' heights, both in Chicago and in other major cities in the U. S., is a topic which many liked.

Longest rivers in the world, lengths of spans of famous bridges in U. S., numbers of stations listed with network broadcasting, rising coffee prices, population of 10 largest cities in U. S., languages spoken in the world, number of TV channels in U. S. —all these topics furnish some material which is different and unusual and develops student motivation.

I offer each student one assigned topic to graph and one free choice topic in working his or her graph. After planning and working out the details, I permit the graphs to be colored or illustrated to add to their effectiveness.

I believe that any graph should tell a story, whether it be a circle graph or a vertical or horizontal bar graph. Students add their own meaning to their graphs by drawing figures or other models associated with their measurements.

This seems to make their work more enjoyable, more meaningful, and they do not want to stop with only two graphs once they get started.

[2] Leo J. Brueckner, Foster E. Grossnickle, Elda L. Merton, *Thinking with Numbers* (Philadelphia: John C. Winston Co.).

Ninth-grade general mathematics

by Thomas E. Waddill, Green City District R-1, Green City, Missouri

Let us assume here that there are five main items to be considered in teaching this course:

1. Basic understandings and abilities required of the competent citizen;
2. Knowledge both desirable and useful which we would like to teach if the time allotted and ability of the pupils will permit;
3. Knowledge basic to success in future mathematics courses;
4. Guidance and motivation;
5. Providing for individual differences.

Since the content of this course must be variable to fit the needs and abilities of different school populations, and because it is often the first course in high-school mathematics and is required of many ninth-grade pupils, it may very well be the most important mathematics course offered and certainly the most difficult to teach.

Some means must be provided for considerable review of the fundamental operations in whole numbers, common and decimal fractions, per cents, and measurement. The first of the year is a good time to provide for a general review as it will give the pupil a sense of security and will also give the teacher an opportunity to diagnose the various individual difficulties. This review must not be continued in such a way as to permit the pupils to feel that "we've already had all this."

A method that I have found fairly successful is to use two basic textbooks; one for the basic part of the course and the other for arithmetic review, drill, and applications. In the latter text[1] inventory tests are set up in such a way that as soon as a pupil knows which problems he missed, he knows immediately which chapters to turn to for study and additional problems. These inventories tell the pupil his individual needs and the results help the teacher direct review assignments. I have found that a great deal of review is helpful to every pupil because of (1) increased understanding, (2) increased accuracy, or (3) both. I rely a great deal on the one text for drill material, diagnostic testing, practical applications, individual assignments, and optional assignments. This text contains ample material and is organized to provide for individual differences and for the basic understandings needed by the competent citizen.

The other text, *Mathematics, A First Course*, by Rosskopf, Aten, and Reeve,[2] is used simultaneously with the other text. This text provides plenty of material for the study of geometry, algebra, business arithmetic and trigonometry, along with many applications. With high-ability groups who need little arithmetic review, you can supplement this text with additional material on geometry and algebra. It also provides for some arithmetic review and also provides for individual differences.

I have found it successful to have a number of concurrent assignments:

1. An arithmetic review assignment from Stein required of everyone, possibly allowing six weeks for its completion.
2. Daily or weekly assignments on geometry from Rosskopf.
3. Optional assignments:
   a. Reports—men of mathematics, historical notes, recreation, uses of mathematics, etc.
   b. Designs—geometric, automobiles, airplanes, etc.

[1] Edwin I. Stein, *Refresher Arithmetic with Practical Applications* (New York: Allyn and Bacon, 1948).

[2] Myron Rosskopf, Harold Aten, William Reeve, *Mathematics, A First Course* (New York: McGraw-Hill Book Co., Inc., 1951).

c. Problems more difficult than those required.
d. Scrapbooks or other projects.

These assignments are correlated to supplement each other; to provide remedial and maintenance work with proper attention to order of presentation. Some of the optional assignments are suggested to supplement the particular topic being studied while others may be given for guidance and motivation at any time. There are several good pamphlets made available by industry and business. Their attractive appearances appeal to the ninth-grade pupil and they also provide discussion material and suggestions for further study and projects. Any interest a pupil may have, if encouraged and guided properly, may be directed toward the cultural, esthetic, and practical values of mathematics for his daily and future life.

In summary, this method implies that we teach as many of the fundamentals of algebra, geometry, and trigonometry as possible, while providing arithmetic for those who need it and for those who desire to improve.

Why pupils elect to take mathematics[1]

by Roy G. Long, Bosse High School, Evansville, Indiana

For several semesters at Bosse High School, the students in many of the mathematics classes have been required to carry out a project on some subject of math as an individual and contribute his results to the class. These projects are used in arithmetic, algebra, and geometry classes as an extra credit incentive for the students. The list of topics is usually developed by teacher-pupil discussions, the students being encouraged to originate their own projects to fit in with what the class is studying. After the student has selected his topic, a conference is held with the teacher to discuss plans for developing the project. The projects are done by the student on his own time and must be completed by a specified time.

In the subject of algebra, the projects so far have been confined to the advanced algebra classes, but an attempt is being made to include some in the second semester of beginning algebra. Some very worthwhile algebra projects of the past have included:

a. Models of the conic sections
b. Models of three-dimensional graphs of three unknowns
c. Models of $(a+b)^2$ and $(a+b)^3$
d. Class demonstrations on maximum and minimum surfaces with soap bubbles
e. Posters on various subjects such as uses of parabolas, circles, and ellipses
f. Graphs of trigonometric functions, conic sections, and types of graphs
g. Charts showing derivation and meaning of words in algebra
h. Systems of balances showing solutions of equations by use of the axioms

In the arithmetic classes, the projects have been centered around the importance of arithmetic and its applications. The subjects of these projects are usually coordinated with the assignments on that particular topic so that the project will be timely. Some subjects of projects used last semester included the following:

a. Arithmetic in the newspapers and magazines

[1] From the *Indiana Mathematics Newsletter*, April 1954.

b. Household utilities—student visited local public utilities, obtained rates, and explained their use to the class
c. Federal and state income taxes—student obtained forms and information on these taxes and explained their use to class
d. Local property taxes—student obtained tax rates and explained how tax rates are determined and used
e. Arithmetic in insurance of all kinds
f. A visit to a local bank arranged for the class, display of banking materials
g. Information on installment buying
h. Complete study of all uses of arithmetic around the home
i. Use of arithmetic in games and sports
j. Scale drawings of original house plans
k. Construction of puzzles
l. Comparison of several number systems
m. An original playlet, radio program, or quiz show with arithmetic ideas
n. Displays of commonly used units of measures

In the geometry classes the projects are usually centered around models, as almost any theorem may be made into a model. The uses of geometry have been used as ideas for posters and displays. Some of the more recent projects have included:

a. Models of the theorem of Pythagoras
b. Circle board to demonstrate all theorems on circles
c. Angle trisection devices and their proof
d. Models of plane geometry theorems using eyelet punch and eyelets with thin strips of fiber board
e. Original geometric designs on cloth, cardboard and transparencies
f. Posters on the uses of geometry in carpentry, sports, church design, home, farming, transportation, and architecture
g. Posters on optical illusions
h. Root meanings of words with meter endings
i. Construction of models of solid geometry theorems
j. Construction of geometric puzzles and games

Much has been written concerning the objectives and values of project work and it is probably still a controversial issue. However, the objectives that have been achieved with our projects can be summarized in these points:

a. The student's interest in mathematics is elevated to a higher plane.
b. Each student feels a part of the whole as he contributes his part.
c. Understandings of certain fundamental theorems of mathematics are improved by visual means.
d. The many important uses of mathematics are emphasized.
e. The mathematics classroom becomes an interesting and inviting room.

The students have displayed a keen interest in doing projects and often ask to do several during the semester. To a number of students, the sense appeal of projects is very valuable. Some of the seemingly less talented in classroom work have been the most talented in construction of a model or a poster. Anything that we teachers can do to promote a better taste for mathematics in our students' minds will lead to better understandings of mathematics. The enrollment in these classes has constantly increased at Bosse since World War II, though the emphasis on mathematics has somewhat decreased nationwide. Just as young people consume the foods most delectable to their tastes, the students consume the things they find most delightful in their classes. We feel that these projects have had a small but important part in making the food for thought in mathematics more interesting to the students.

Your professional dates

The information below gives the date, name, and place of meeting, with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street N.W., Washington 6, D. C.

NCTM convention dates

April 13–16, 1955
ANNUAL MEETING
Statler Hotel, Boston, Massachusetts
Jackson Adkins, local chairman, Phillips Exeter Academy, Exeter, New Hampshire

July 4, 1955
JOINT MEETING WITH NEA
Chicago, Illinois
E. H. C. Hildebrandt, local chairman, Northwestern University, Evanston, Illinois

August 21–24, 1955
SUMMER MEETING
Indiana University, Bloomington, Indiana
Philip Peak, local chairman, Indiana University, Bloomington, Indiana

December 27–30, 1955
CHRISTMAS MEETING
Sheraton-Park Hotel, Washington, D. C.
Veryl Schult, local chairman, Wilson Teachers College, Washington 9, D. C.

Other professional dates

Annual Spring Conference of the Minnesota Council of Mathematics Teachers
April 29–30, 1955
Coffman Memorial, University of Minnesota, Minneapolis, Minnesota
Miss Angela Untereker, Technical High School, St. Cloud, Minnesota

Spring Meeting of the Colorado Council of Teachers of Mathematics
April 30, 1955
Pueblo Junior College, Pueblo, Colorado
Carl Wilkerson, 129 Van Buren, Pueblo, Colorado

University of Oklahoma Institute for Teachers of Mathematics
June 6–17, 1955
University of Oklahoma, Norman, Oklahoma
J. O. Hassler, University of Oklahoma, Norman, Oklahoma

National Science Foundation Collegiate Conference
June 13–July 22, 1955
Oklahoma A. and M. College, Stillwater, Oklahoma
L. Wayne Johnson, Oklahoma A. and M. College, Stillwater, Oklahoma

Sixth Annual Mathematics Institute
June 19–24, 1955
Louisiana State University, Baton Rouge, Louisiana
Houston T. Karnes, Louisiana State University, Baton Rouge 3, Louisiana

Workshop on Elementary Mathematics
June 20–July 1, 1955
Iowa State Teachers College, Cedar Falls, Iowa
I. H. Brune, Iowa State Teachers College, Cedar Falls, Iowa

Conference on Secondary Mathematics
June 20–July 1, 1955
Iowa State Teachers College, Cedar Falls, Iowa
H. Van Engen, Iowa State Teachers College, Cedar Falls, Iowa

Eighth Annual Workshop for Teachers of Mathematics
June 20–July 2, 1955
Indiana University, Bloomington, Indiana
Philip Peak, School of Education, Indiana University, Bloomington, Indiana

California Conference for Teachers of Mathematics
July 5–15, 1955
University of California, Los Angeles, California
Clifford Bell, University of California, Los Angeles 24, California

Third New Jersey Mathematics Institute
July 6–15, 1955
Rutgers University, New Brunswick, New Jersey
Director of the Summer Session, Rutgers University, New Brunswick, New Jersey

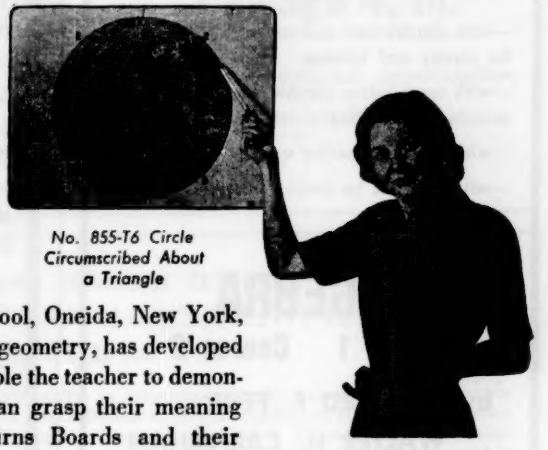
Third Annual Mathematics Institute of the Florida Council of Teachers of Mathematics
August 18–20, 1955
University of Florida, Gainesville, Florida
Kenneth P. Kidd, University of Florida, Gainesville, Florida

Seventh Annual Mathematics Institute of the Association of Teachers of Mathematics in New England
August 18–25, 1955
Middlebury College, Middlebury, Vermont
Harriet Howard, Ethel Walker School, Simsbury, Connecticut

THE BURNS BOARDS

by Frances M. Burns

A series of manipulative sensory aids for more effective learning of INTUITIVE and DEMONSTRATIVE GEOMETRY at the junior and senior high school levels



No. 855-T6 Circle Circumscribed About a Triangle

Frances M. Burns, of Oneida High School, Oneida, New York, well-known authority on the teaching of geometry, has developed a series of manipulative devices that enable the teacher to demonstrate basic principles so that pupils can grasp their meaning quickly. The ready acceptance of Burns Boards and their rapidly growing popularity with educators are due to many advantages, including the following:

- Burns Boards create interest in geometry, without which little effective learning can be accomplished.
- Burns Boards give meaning to geometric principles, propositions and definitions.
- Burns Boards allow correct training in induction.
- Burns Boards lead the pupil to discover for himself relationships of geometric patterns to one another.
- Burns Boards help the student to generalize and to state the generalization in concise language.
- Burns Boards save the time of both the teacher and the students.

BURNS BOARDS Teacher Models

10 large models (18" high x 24" wide) for classroom demonstrations by the teacher. Each board is equipped with elastics and pegs for quick and easy manipulation. The teacher's instructions include examples which can be quickly set up on the demonstration boards. As the teacher demonstrates theorems, each pupil with his own board can work out the same problems with her.

No. 855 Per set $25.00

BURNS PUPILS' BOARDS

A true laboratory approach to geometry is assured through the use of Burns Pupils' Boards. They encourage student investigation and discovery. Classroom experience has shown them to be a powerful stimulus to learning. It is suggested for best results that students work in pairs with one board to the pair. Pupil models ¼" cardboard (9" × 12") are complete with elastics and pegs. Full directions for use and student work sheet is furnished with each board. Additional work sheets may be purchased in pads of 20.

No. 856 Per set of 10 $5.00

Ask Your Dealer for Full Details!

IDEAL SCHOOL SUPPLY CO.

8312 South Birkhoff Avenue, Chicago 20, Ill.